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Rearrangements and nonlinear analysis of vortices

Douglas, Robert

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Robert Douglas

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R. J. Douglas

R.J. Douglas

Abstract

For a given function (in an L^p space), the set of rearrangements is the set of functions whose super level sets have the same measure as those of the original function. This thesis presents a characterisation of the weak closure of the set of rearrangements for a given function defined on an unbounded domain. In Chapter 2 we characterise the weak closure of the set of rearrangements for non–negative L^p functions defined on the half–line, where $1 < p < \infty$, and show that the set of extreme points of this convex weakly compact set coincides with the set of rearrangements of curtailments. Chapter 3 uses the concept of measure preserving transformations to extend this work to non–negative L^p functions (p as above) defined on unbounded domains of $\mathbb{R}^n$. Chapter 4 describes to what extent we may extend our characterisation of the weak closure of the set of rearrangements to non–negative L^1 functions defined on an unbounded domain.

An equivalent characterisation of the weak closure for non–negative L^p functions ($1 < p < \infty$) defined on the half–line or the real line is given in Chapter 5. Chapters 6, 7 and 8 are concerned with the use of earlier work to study the maximising sequences of two variational problems for steady vortices. The variational problems are shown to attain maxima relative to the weak closure of the set of rearrangements, and we establish some properties of the maximising functions. Each maximiser satisfies a partial differential equation for the stream function of a steady flow.

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Contents

Chapter 1

Introduction

1.1 Sets of Rearrangements

This thesis studies extreme points of the weak closure of the set of rearrangements of a function on an unbounded domain, and gives two applications to steady vortices. Two non–negative measurable functions f and g, defined on a subset of $\mathbf{R}^n$, are *rearrangements* if

$$\lambda\left(f^{-1}[\beta,\infty)\right) = \lambda\left(g^{-1}[\beta,\infty)\right) \tag{1.1}$$

for all positive β, where λ is a σ–finite measure absolutely continuous with respect to n–dimensional Lebesgue measure. For a fixed non–negative function f_0, we write $R(f_0)$ to denote the set of rearrangements of f_0. Eydeland, Spruck and Turkington [**17**] gave the following characterisation of the set of rearrangements for f_0 defined on the half–line with properties as above,

$$R(f_0) = \left\{ w \geq 0, w \text{ measurable} \middle| \int_0^\infty (w-\sigma)_+ = \int_0^\infty (f_0-\sigma)_+, \forall \sigma > 0 \right\} \tag{1.2}$$

where h_+ denotes the positive part of h.

It is natural for certain variational problems to have the set of rearrangements of a fixed function as the constraint set. The theory of steady vortex rings in an ideal fluid provides an example (we study this problem later). The fact that the constraint set is a set of rearrangements can compensate for the loss of compactness in a variational problem on an unbounded domain, yielding maximising (minimising) sequences with the required convergence properties, by reducing a problem on an unbounded domain to one on a bounded domain, as in [**3**]. Our approach however is to work directly on unbounded domains.

This thesis characterises the weak closure of the set of rearrangements of a fixed non–negative L^p function (for $1 < p < \infty$) defined on an open unbounded domain of infinite measure, and identifies the extreme points. This set is shown to be convex and weakly compact. In the latter part of the thesis we apply the earlier results to the study of maximising sequences of two variational problems for steady vortices. Maximisers relative to the weak closure of the set of rearrangements are shown to exist, and each satisfies a partial differential equation for the stream function of a steady flow. In addition we show that the maximisers have bounded support.

1.2 Previous Research

We give a brief history of research relevant to this thesis. Establishing properties of the set of rearrangements of a fixed function, and its weak closure, has long been of interest. Ryff [**30**] showed that for non–negative L^1 functions defined on the unit interval, the weak closure of the set of rearrangements is a convex set (whence it is equal to the closed convex hull of the set of rearrangements). He found a characterisation for this set in terms of integrals of decreasing rearrangements of its elements. Brown [**6**] extended this work by showing convexity of the weak closure of the set of rearrangements for non–negative L^p functions (for $1 < p < \infty$)

defined on the unit interval. The aforementioned authors made use of doubly stochastic operators. Their results were rediscovered by Burton [**7**] using a direct proof. Burton extended the result to non–negative L^p functions defined on a domain of finite measure. He noted that the set is weakly sequentially compact (this follows easily for $1 < p < \infty$, and by the Dunford–Pettis criterion [**15**, Theorem 1.3] for weak compactness in L^1). Burton and Ryan [**10**] characterised the weak closure of the set of rearrangements for non–negative L^p (for $1 \leq p < \infty$) functions defined on the unit interval. They improved previous results by showing that the intersection of the weak closure of the set of rearrangements with a set of finitely many linear constraints is equal to the closed convex hull of the set of rearrangements intersected with the constraint set. The constraint set, a subset of L^p, fixes the inner product with respect to finitely many given L^q functions (where q denotes the conjugate exponent of p). Burton and Ryan used the above result to show that for a bounded linear operator $T : L^p(I) \to \mathbf{R}^n$, where $1 \leq p < \infty$ and I denotes the unit interval, we have

$$T\left(R(f_0)\right) = T\left(\overline{R(f_0)}^w\right). \tag{1.3}$$

They deduced $T\left(R(f_0)\right)$ is compact and convex. This shows that all finite dimensional projections of a set of rearrangements are convex.

Inequalities involving rearrangements of functions have long been studied. For non–negative measurable functions f, g and h defined on the real line, we state Riesz's inequality

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(x-y)h(y)dxdy \leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^{\triangle}(x)g^{\triangle}(x-y)h^{\triangle}(y)dxdy \tag{1.4}$$

where $f^{\triangle}$ denotes the symmetric decreasing rearrangement of f. This result was first proved by F. Riesz [**26**]: another proof was given by Hardy, Littlewood and Polya [**22**, pages 279–287]. Friedman and McLeod [**19**] considered when the inequality in (1.4) is strict. Assuming that g is symmetrically decreasing, they showed if equality holds in (1.4), then $f = f^{\triangle}$ and $h = h^{\triangle}$ for some suitable choice of origin on the real axis. Note that the integrals in (1.4) are unchanged if f and h are translated by the same amount. Brascamp, Lieb and Luttinger [**4**] proved a generalised form of Riesz's inequality; for non–negative measurable functions f_j defined on $\mathbf{R}$, and real numbers a_{jm}, where $j = 1, ..., k$ and $m = 1, ..., n$, we have

$$\int_{\mathbf{R}^n} \prod_{j=1}^{k} f_j \left(\sum_{m=1}^{n} a_{jm} x_m \right) d^n x \leq \int_{\mathbf{R}^n} \prod_{j=1}^{k} f_j^{\triangle} \left(\sum_{m=1}^{n} a_{jm} x_m \right) d^n x. \tag{1.5}$$

We now consider inequalities involving norms of decreasing rearrangements on function spaces. Crowe, Zweibel and Rosenbloom [**13**] show that the operator which maps a function (defined on $\mathbf{R}$) to its symmetric decreasing rearrangement is a contraction on L^p for $1 \leq p < \infty$. Moreover, Epperson [**16**] was able to show that the operator is non–expansive in certain Orlicz spaces. Let u be a non–negative measurable function defined on $\mathbf{R}^n$, and suppose u has compact support. We define the spherically decreasing rearrangement u^* by

$$u^*(x) = \sup \{t | \mu\{x | u(t) > t\} > \alpha(n) |x|^n\} \tag{1.6}$$

where $\alpha(n)$ is the volume of the unit ball in $\mathbf{R}^n$, and μ denotes Lebesgue measure. If in addition $u \in W^{1,p}(\mathbf{R}^n)$ for $1 \leq p < \infty$, Polya and Szego [**25**] showed that

$$\int_{\mathbf{R}^n} |\nabla u^*|^p d\mu \leq \int_{\mathbf{R}^n} |\nabla u|^p d\mu \tag{1.7}$$

Brothers and Ziemer [**5**] considered in what circumstances equality holds in (1.7). They showed that if the distribution function of u is absolutely continuous, and equality holds in (1.7), then u is a translate of u^*. They generalised this result to integral inequalities of the form

$$\int_{\mathbb{R}^n} A(|\nabla u^*|)d\mu \leq \int_{\mathbb{R}^n} A(|\nabla u|)d\mu \tag{1.8}$$

where $A : [0,\infty) \to [0,\infty)$ is twice continuously differentiable, $A(0) = 0$, and $A^{\frac{1}{p}}$ is convex, by establishing certain conditions under which inequality is strict in (1.8) unless u is a translate of u^*. An inequality analogous to (1.7) was found for the Steiner symmetrisation of a non–negative function defined on the half–plane, by Fraenkel and Berger [**18**, Appendix 1]. We defer further discussion of this result until Chapter 6.

For non–negative measurable functions f and g defined on the half–line, the following inequality is well known:

$$\int_0^\infty fg \leq \int_0^\infty f^* g^* \tag{1.9}$$

where f^* denotes the decreasing rearrangement of f. (We require $\int_0^\infty fg$ to be finite for (1.9) to be meaningful). This result is presented in [**22**]. Burton [**7**] extended this result to functions defined on a finite measure space. For simplicity we state Burton's results for non–negative functions defined on the unit interval (which we denote I). For $f_0 \in L^p(I)$, $g_0 \in L^q(I)$, where $1 \leq p \leq \infty$, and q denotes the conjugate exponent of p, we have

$$\int_0^1 fg \leq \int_0^1 f_0^* g_0^* \tag{1.10}$$

for all $f \in R(f_0)$, $g \in R(g_0)$. [**7**, Lemma 3] shows that if there exists $\tilde{f} \in R(f_0)$ such that $\tilde{f} = \varphi \circ g_0$ almost everywhere for some increasing function φ, then equality holds in (1.10) for $\tilde{f} \in R(f_0)$, $g_0 \in R(g_0)$.

We now consider maximisation of functionals over sets of rearrangements. For non–negative $f_0 \in L^p(I)$, $g \in L^q(I)$, where $1 \leq p < \infty$, and q denotes the conjugate exponent of p, define $\Psi(h) = \int_0^1 hg$, for $h \in L^p(I)$. If there exists $\tilde{f} \in R(f_0)$ such that $\tilde{f} = \varphi \circ g$ almost everywhere for some increasing function φ, [**7**, Theorem 3] states that $\tilde{f}$ is the unique maximiser of Ψ relative to the weak closure of $R(f_0)$. For non–atomic separable measure spaces, [**7**, Theorem 5] yields a partial converse; If Ψ has a unique maximiser $\tilde{f}$ relative to $R(f_0)$, then $\tilde{f} = \varphi \circ g$ almost everywhere for some increasing function φ. This result has been used successfully in the study of certain partial differential equations.

Burton [**7**, Theorem 9] showed the existence of weak solutions of free boundary problems for certain semilinear elliptic equations. For simplicity we state these results for the Laplacian, which we denote $\triangle$. Let Ω be a domain in $\mathbf{R}^n$ of finite μ–measure, where μ is equivalent to Lebesgue measure on Ω. Let $1 \leq p < \infty$, and let q denote the conjugate exponent of p. Suppose there exists a compact positive symmetric operator $K : L^p(\Omega) \to L^q(\Omega)$ such that $-\triangle Ku = u$ almost everywhere in Ω for all $u \in L^p(\Omega)$. Further suppose that there exists $v \in L^q(\Omega) \bigcap W^{2,1}_{loc}(\Omega)$ such that $-\triangle v = 0$ almost everywhere in Ω. Let non–negative $f_0 \in L^p(\Omega)$, for p as above. For $\alpha \in \mathbf{R}$, if there exist $f_1, f_2 \in R(f_0)$ such that

$$\int_\Omega f_1 v d\mu < \alpha < \int_\Omega f_2 v d\mu \tag{1.11}$$

then the functional

$$\Psi(f) = \frac{1}{2} \int_\Omega f K f d\mu \tag{1.12}$$

attains a maximum relative to

$$\left\{ f \in R(f_0) \middle| \int_\Omega f v d\mu = \alpha \right\}. \tag{1.13}$$

Further, for a maximiser $\tilde{f}$, $\tilde{u} = K\tilde{f}$ satisfies

$$-\triangle \tilde{u} = \varphi \circ (\tilde{u} - \lambda v) \tag{1.14}$$

almost everywhere in Ω for some increasing function φ, and some real λ.

A variant of the above result was proved in Burton [**8**, Lemma 2.15]. To avoid complicated notation, we state the result for the Laplacian. For Ω, μ, p as before, let non-negative $f_0 \in L^p(\Omega)$, and let $g \in L^q(\Omega) \bigcap W^{2,1}_{loc}(\Omega)$. Suppose f^* maximises $\int_\Omega fg$ relative to $f \in \overline{R(f_0)}^w$ (we do not require f^* to be the unique maximiser), and that $-\Delta g \geq f^*$ almost everywhere in Ω. Then $f^* \in R(f_0)$, and $f^* = \varphi \circ g$ almost everywhere, for some increasing function φ.

We now discuss maximisation of an energy functional of the same form as Ψ above, but now we maximise over the set of rearrangements of a two-signed function. Let Ω be a non-empty bounded open set in $\mathbf{R}^n$, and let μ denote Lebesgue measure on Ω. For $s_n \leq p < \infty$, where

$$s_n = \begin{cases} 1 & \text{if } n = 1 \\ 2n/(n+2) & \text{if } n \geq 2 \end{cases}$$

let $K : L^p(\Omega) \to H_0^1(\Omega)$ be the inverse for $-\Delta$ with zero Dirichlet boundary conditions (the existence and properties of K are established in [**9**]). Define

$$E(v) = \frac{1}{2} \int_\Omega vKv d\mu \tag{1.15}$$

for $v \in L^p(\Omega)$. Burton and McLeod [**11**, Theorem 2.1] showed the following for (possibly) two-signed $f_0 \in L^p(\Omega)$:

(i) $\inf_{f \in R(f_0)} E(f) = \inf_{f \in \overline{R(f_0)}^w} E(f)$, and the infimum is attained by exactly one element $\tilde{f} \in \overline{R(f_0)}^w$.

(ii) For some decreasing function ϕ, $\tilde{f} = \phi \circ K\tilde{f}$ almost everywhere in Ω. No other rearrangement of $\tilde{f}$ has this property.

(iii) $\tilde{f}$ is essentially one-signed.

(iv) If f_0 is one-signed, $\tilde{f} \in R(f_0)$.

(v) If f_0 is essentially two-signed, the minimiser $\tilde{f}$ is in general not a member of $R(f_0)$. [**11**, Theorem 2.1] gives properties satisfied by the minimisers.

The results stated above are relevant to the theory of steady vortices in an ideal fluid. Benjamin [**3**] proposed a variational problem to seek solutions to the boundary value problem for a steady vortex ring in an ideal fluid flowing along an infinite pipe of circular cross–section. He considered the energy of a perturbation of the fluid. His idea was to seek extremals of the energy relative to the set of rearrangements of a fixed function. The stream function $u - \lambda r^2/2$ gives rise to an axisymmetric velocity v whose cylindrical co–ordinates (r, θ, z) are given by $(-\frac{1}{r}u_z, 0, \frac{1}{r}u_r - \lambda)$. We define the scalar vorticity field ξ by

$$\text{curl } v = (0, \xi, 0). \tag{1.16}$$

We define

$$\mathcal{L}u = -\frac{1}{r}\left(\frac{1}{r}u_r\right)_r - \frac{1}{r^2}u_{zz} \tag{1.17}$$

to obtain

$$\xi = r\mathcal{L}u. \tag{1.18}$$

The region $\xi^{-1}(0, \infty)$ is known as the *vortex core*. We prescribe the fluid speed at infinity relative to the vortex core a fixed value $\lambda > 0$. The measures of the sets $(\xi/r)^{-1}[\alpha, \infty)$, where $\alpha > 0$, are preserved in all axisymmetric (including unsteady) flows of an ideal fluid in $\mathbf{R}^3$. Benjamin's approach seeks a solution for which ξ/r is a rearrangement of a prescribed function f_0. Let $K : L^p \to H_0^1$ denote the inverse for $\mathcal{L}$ with zero Dirichlet boundary conditions. (The existence and properties of K are established in [**9**].) Let non–negative non–zero $f_0 \in L^p$ (for $p > 5$) have bounded support, and let the value of λ be sufficiently small. Burton [**9**] showed the existence of $\tilde{f} \in R(f_0)$ such that $\tilde{u} = K\tilde{f}$ satisfies

$$\mathcal{L}\tilde{u} = \varphi(\tilde{u} - \frac{1}{2}\lambda y^2) \tag{1.19}$$

almost everywhere for some increasing function φ. (1.19) is a partial differential equation for the stream function of a steady flow.

Finally, we consider a variational problem where rearrangements techniques yield a minimising sequence with the required properties. Lieb [**23**] sought to show the existence of a unique minimiser for the functional

$$\epsilon(\phi) = \int_{\mathbb{R}^3} |\nabla\phi(x)|^2 \, dx - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |\phi(x)|^2 \, |x-y|^{-1} \, |\phi(y)|^2 \, dxdy \tag{1.20}$$

relative to $E(\lambda) = \inf\{\epsilon(\phi) | \phi \in W^{1,2}(\mathbb{R}^3), ||\phi||_2 \leq \lambda\}$. ϵ is not a convex function, precluding the usual methods to show existence and uniqueness of the minimum. Let ϕ^* denote the spherically decreasing rearrangement of ϕ. Lieb showed that for ϕ not equal to a translate of ϕ^*, $\epsilon(\phi^*) < \epsilon(\phi)$. As part of his proof, he gave a neat argument showing that $||\nabla\phi||_2 \geq ||\nabla\phi^*||_2$ for ϕ as above. Lieb established the existence and uniqueness (modulo translation) of a minimising ϕ.

1.3 Outline of Thesis

We study the weak closure of the set of rearrangements for a non-negative L^p function (for $1 < p < \infty$) defined on the half-line in Chapter 2. This convex set has extreme points which are not rearrangements of the original function: this contrasts with the result for a function defined on a domain of finite measure. For a given function defined on the half-line, a *curtailment* at l (where l is an extended real number) is a function equal to the decreasing rearrangement of the original function on the interval $(0, l)$, and zero elsewhere. The set of extreme points of the weak closure of the set of rearrangements coincides with the set of rearrangements of curtailments of the original function. We show that the weak closure of the set of rearrangements is convex and weakly (sequentially) compact, and we give a characterisation of this set.

Chapter 3 extends the work of the previous chapter to L^p functions (for $1 < p < \infty$) defined on open unbounded domains of infinite measure. We establish the existence of a measure preserving transformation between the unbounded domain and the half–line, and by considering the equivalent problem on the half–line, we show that the weak closure of the set of rearrangements is convex and weakly (sequentially) compact. The set of extreme points of this set coincides with the set of rearrangements of curtailments.

For a non–negative L^1 function defined on an open unbounded domain of infinite measure, the weak closure of the set of rearrangements is not weakly sequentially compact. However, in Chapter 4, we show that this set is convex, whence it is equal to the closed convex hull of the set of rearrangements. We specify a closed, bounded and convex superset for the weak closure of the set of rearrangements, the form of which gives some insight into the geometry of the set.

In [**10**, Lemma 5], the weak closure of the set of rearrangements for an L^p function f_0 (for $1 \leq p < \infty$) defined on the unit interval was shown to be equal to the set of non–negative functions w satisfying $||w||_1 = ||f_0||_1$ and $\int_0^s w^* \leq \int_0^s f_0^*$ for each $s \in (0, 1)$, where w^* denotes the decreasing rearrangement of w. We show that we may extend the characterisation to non–negative L^p functions (for $1 < p < \infty$) defined on the half–line by discarding the condition that the 1–norms are equal. This set is shown to be equal to the characterisation of the weak closure of the set of rearrangements discussed in Chapter 2.

Chapter 6 considers a variational problem, proposed by Benjamin [**3**], arising from the theory of steady vortex rings in a 3–dimensional ideal fluid. For sufficiently small values of a parameter, Burton [**9**] showed the existence of maximisers relative to the set of rearrangements of a prescribed function with bounded support. However for large values of the parameter there is no maximiser relative to the set of rearrangements. For a pre–assigned function (possibly with unbounded support) we show the variational functional attains a maximum relative to the weak closure of the set of rearrangements for all positive values of the parameter, and that the maximisers belong to the set of rearrangements of curtailments. We investigate further properties of the maximising functions in Chapter 7. Each maximiser is shown to be the solution (in the weak sense) of a partial differential equation for the stream function of a steady flow. Each has bounded support. We show that the supremum of the variational functional relative to the set of rearrangements is equal to the supremum relative to the weak closure of the set of rearrangements, a non–obvious fact because the functional is not weakly continuous, or even weakly upper semicontinuous.

Chapter 8 considers a variational problem arising from the theory of steady vortex pairs in a planar ideal fluid. We show that maximisers exist relative to the weak closure of the set of rearrangements of a prescribed function, for all positive values of a parameter. We establish that the maximisers are extreme points, whence they belong to the set of rearrangements of curtailments, and that they have bounded support. Each maximiser satisfies the partial differential equation for the stream function of a steady flow in an infinite channel. For a given maximiser we can choose a sequence of rearrangements of the pre–assigned function such that the maximiser is the weak limit of the sequence, and that the functional values of the sequence converge to that of the maximiser. Some of the proofs in this chapter are similar to those given in Chapters 6 and 7, but they are included for completeness.

Chapter 2

Extreme points of the weak closure of the set of rearrangements on the half line

2.1 Introduction

In this chapter we prove the following theorem:

Theorem 1 Let non-negative $f_0 \in L^p(0,\infty)$, $1 < p < \infty$. Let μ denote one-dimensional Lebesgue measure. Let $h_1, ..., h_n \in L^q(0,\infty)$ (where q denotes the conjugate exponent of p) and let $\alpha_1, ..., \alpha_n \in \mathbf{R}$. Define

$$W = \{w \geq 0, w \text{ measurable} | \int_0^\infty (w-\sigma)_+ d\mu \leq \int_0^\infty (f_0 - \sigma)_+ d\mu, \forall \sigma > 0\} \quad (2.1)$$

$$G = \{w \in L^p(0,\infty) | \int_0^\infty w h_i d\mu = \alpha_i, i = 1, ..., n\} \quad (2.2)$$

Then the following is true:

(i) W is a convex,weakly compact subset of $L^p(0,\infty)$.

(ii) extr $W \bigcap G = RC(f_0) \bigcap G$.

(iii) $\overline{\text{conv}RC(f_0) \bigcap G} = W \bigcap G$.

(iv) $W = \overline{\text{conv}R(f_0)} = \overline{R(f_0)}^w$.

h_+ denotes the non-negative part of h, R(f_0) denotes the set of rearrangements of f_0, and extr A, $\overline{A}^w$ denote respectively the extreme points of A, and the weak closure of A. RC(f_0) denotes the set of rearrangements of curtailments of f_0: this set will be defined subsequently.

In Burton and Ryan [**10**, Lemma 5], it was shown that

$$\overline{R(v)}^w = \overline{\text{conv}R(v)} \tag{2.3}$$

for non-negative $v \in L^p(I)$, $1 \leq p \leq \infty$, where I denotes the unit interval. Furthermore it was shown that

$$\overline{R(v)}^w = \{w \geq 0 | \int_0^s w^* \leq \int_0^s v^*, 0 < s < 1, ||w||_1 = ||v||_1\} \tag{2.4}$$

and

$$\text{extr } \overline{R(v)}^w = R(v) \tag{2.5}$$

where w^* denotes the decreasing rearrangement of w. The above theorem extends this work to functions defined on the half line, but does not make use of a set of the form of (2.4). We consider to what extent (2.4) may characterise the weak closure of the set of rearrangements for a function defined on the half–line in Chapter 5.

The Banach–Alaoglu Theorem is used to obtain (i). For (ii), we calculate the extreme points of $W \bigcap G$, and apply the Krein–Milman Theorem to obtain (iii). For non–zero f_0 as in Theorem 1, there are extreme points of $\overline{R(f_0)}^w$ which are not rearrangements, unlike the analogous bounded domain result proved by

Burton and Ryan. The set of extreme points coincides with the set of rearrangements of curtailments of f_0, which is defined in the next section. The proof is completed by showing that any member of $RC(f_0)$ belongs to the weak sequential closure of $R(f_0)$. The form of W, based on an equivalent definition of the set of rearrangements by Eydeland, Spruck and Turkington [**17**], enabled the proof to be simpler than a proof using the techniques of Burton and Ryan [**10**].

It may be noted that the method of proof used in this chapter does not extend to non-negative functions $g_0 \in L^1(0,\infty)$, because $\overline{R(g_0)}^w$ does not lie in any weakly compact set unless g_0 is identically zero (see Lemma 7). However it may be shown that $\overline{R(g_0)}^w = \overline{\mathrm{conv} R(g_0)}$ (see Chapter 4).

2.2 Definitions and Notation

Let f,g,h be non–negative Lebesgue measurable functions defined on the half line. Let $\overline{\mathbb{R}}$ denote the extended real numbers, and let μ denote 1–dimensional Lebesgue measure.

Using standard arguments, it was shown by Eydeland, Spruck and Turkington [**17**] that

$$R(f) = \{g \geq 0 | \int_0^\infty (g-\sigma)_+ = \int_0^\infty (f-\sigma)_+, \forall \sigma > 0\} \tag{2.6}$$

where h_+ denotes the non–negative part of h, that is

$$h_+(t) = \begin{cases} h(t) & \text{if } h(t) \geq 0 \\ 0 & \text{if } h(t) < 0 \end{cases}$$

Let f^* denote the decreasing rearrangement of f on $(0,\infty)$. For a set $A \subset (0,\infty)$, we denote by 1_A the indicator function of the set A, that is

$$1_A(t) = \begin{cases} 1 & \text{if } t \in A \\ 0 & \text{if } t \notin A \end{cases}$$

Definition g is a *curtailment* of f at $l \in \overline{\mathbf{R}}$ if

$$g = 1_{(0,l)} f^*. \tag{2.7}$$

We say that g is a *rearrangement of a curtailment* of f if g^* is a curtailment of f at some $l \in \overline{\mathbf{R}}$. *RC(f)* will denote the set of rearrangements of curtailments of f.

2.3 Characterisation of the weak closure of the set of rearrangements

Lemma 1 *Let f_0 be non–negative, $f_0 \in L^p(0,\infty)$,$1 < p < \infty$. Define*

$$W = \{f \geq 0, f \ \mathit{measurable} | \int_0^\infty (f-\sigma)_+ \leq \int_0^\infty (f_0 - \sigma)_+, \forall \sigma > 0\}. \tag{2.8}$$

Then W is closed,convex, and bounded, whence W is weakly compact (by the Banach–Alaoglu Theorem).

Proof To show W is bounded, we define

$$S = \{u | u(s) = \sum_{i=1}^{n} \alpha_i (s - \sigma_i)_+, \text{ for some } 0 \leq \sigma_1 < \ldots < \sigma_n, \alpha_i \geq 0\} \tag{2.9}$$

where $n = n(u)$.

If $f \in W$, then

$$\int_0^\infty u(f) d\lambda \leq \int_0^\infty u(f_0) d\lambda, \ \forall u \in S. \tag{2.10}$$

(If $\sigma_1 = 0$, application of the Monotone Convergence Theorem yields the above result).

It will later be shown that ψ, where $\psi(s) = s^p$ is an increasing pointwise limit of functions in S, the $\{\psi_n\}_{n=1}^{\infty}$ say; consequently for $f \in W$, applying the Monotone Convergence Theorem,

$$\int_0^\infty \psi(f)d\lambda = \int_0^\infty \lim_{n\to\infty} \psi_n(f)d\lambda = \lim_{n\to\infty} \int_0^\infty \psi_n(f)d\lambda \tag{2.11}$$
$$\leq \lim_{n\to\infty} \int_0^\infty \psi_n(f_0)d\lambda \tag{2.12}$$
$$= \int_0^\infty \lim_{n\to\infty} \psi_n(f_0)d\lambda = \int_0^\infty \psi(f_0)d\lambda \tag{2.13}$$

whence

$$\int_0^\infty f^p \leq \int_0^\infty f_0^p d\lambda = ||f_0||_p^p. \tag{2.14}$$

This shows W is bounded.

It remains to show that ψ is an increasing pointwise limit of functions in S. Now $\psi'(s) = ps^{p-1}$ where ψ' denotes the derivative of ψ. ψ' is increasing, therefore by the Fundamental Approximation Lemma, we may choose an increasing sequence $\{\phi_n\}_{n=1}^{\infty}$ of increasing simple functions, with $\phi_n \to \Psi'$ pointwise.

For $0 < s < \infty$,

$$\int_0^s \phi_n d\lambda \to \int_0^s \psi' d\lambda = \psi(s) \tag{2.15}$$

as $n \to \infty$ by the Monotone Convergence Theorem.

Define $\psi_n(s) = \int_0^s \phi_n$ for each $n \in \mathbf{N}$. We may write

$$\phi_n = \sum_{i=1}^{m(n)} \alpha_i 1_{[\sigma_i,\infty)} \tag{2.16}$$

for some $\alpha_i \in \mathbf{R}^+$ (the non–negative reals), and $0 = \sigma_1 < ... < \sigma_{m(n)}$ (because the ϕ_n are monotone increasing).

Thus

$$\psi_n(s) = \sum_{i=1}^{m(n)} \alpha_i (s - \sigma_i)_+ \tag{2.17}$$

so $\psi_n \in S$, $\forall n \in \mathbf{N}$.

Let $\sigma > 0$ be given. Define, for non-negative $f \in L^p(\Omega)$, p as above, the function

$$\Psi_\sigma(f) = \int_0^\infty (f - \sigma)_+ d\lambda \tag{2.18}$$

Then Ψ_σ is convex (immediate) and lower semicontinuous (follows easily by Fatou's Lemma). Thus W is closed and convex. This completes the proof.

We recall some concepts from convex analysis

Definitions

Let V be a vector space, $f : V \to \overline{\mathbf{R}}$ a convex function. f is *proper* if $f(x) > -\infty$ for all $x \in V$ and $f(x_0) < \infty$ for some $x_0 \in V$.

The *effective domain* of f is defined by

$$\text{dom } f = \{x \in V | f(x) < \infty\} \tag{2.19}$$

We quote a theorem which is used in the proof of the next result.

Theorem Every proper convex function on $\mathbf{R}^n$ is continuous on the interior of its effective domain.

Lemma 2 *Let f_0, p and W be as in the statement of Lemma 1. Let $h_1, ..., h_n \in L^q(0,\infty)$ (where q denotes the conjugate exponent of p) and let $\alpha_1, ..., \alpha_n \in \mathbf{R}$. Define*

$$G = \{w \in L^p(0,\infty) | \int_0^\infty w h_i d\lambda = \alpha_i, i = 1, ..., n\}. \tag{2.20}$$

Then $extr(W \bigcap G) \subset RC(f_0) \bigcap G$.

Proof Let $w \in \text{extr}(W \bigcap G)$. Define, for $\sigma \in (0, \infty)$,

$$F(\sigma) = \int_0^\infty (w - \sigma)_+ d\lambda \tag{2.21}$$

$$G(\sigma) = \int_0^\infty (f_0 - \sigma)_+ d\lambda \tag{2.22}$$

$w \in W$, therefore by the definition of W we have $F \leq G$. It is easily verified that F and G are proper, convex functions, whence they are continuous on the interiors of their effective domains, that is $(0, \infty)$.

Define $\Omega = \{\sigma \in (0,\infty) | F(\sigma) < G(\sigma)\}$. This is an open set in $(0,\infty)$, (since F and G are continuous), whence it is a countable union of disjoint open intervals. We first show $w^{-1}(\Omega)$ has zero measure. We suppose not and seek a contradiction. Let (s_0, s_1) be a connected component of Ω, possibly unbounded, with $\mu\{w^{-1}(s_0, s_1)\} > 0$. We establish the existence of $\tau, \epsilon > 0$ such that

$$0 \leq s_0 < \tau - \epsilon < \tau + \epsilon < s_1 \leq \infty \tag{2.23}$$

$$\infty > \mu\{w^{-1}(\tau - \frac{\epsilon}{3}, \tau + \frac{\epsilon}{3})\} > 0 \tag{2.24}$$

Now

$$w^{-1}(s_0, s_1) = \bigcup_{n \in \mathbf{N}} \bigcup_{\tau \in Q_n^*} w^{-1}(\tau - \frac{1}{3n}, \tau + \frac{1}{3n}) \tag{2.25}$$

where $Q_n^* = \mathbf{Q} \bigcap [s_0 + \frac{1}{n}, s_1 - \frac{1}{n}]$. Using countable sub–additivity of the measure, we obtain

$$\sum_{n \in \mathbf{N}} \sum_{\tau \in Q_n^*} \mu\{w^{-1}(\tau - \frac{1}{3n}, \tau + \frac{1}{3n})\} > 0 \tag{2.26}$$

The result follows.

Write

$$E = w^{-1}(\tau - \frac{\epsilon}{3}, \tau + \frac{\epsilon}{3}) \tag{2.27}$$

Then by Lyapunov's Theorem, Cesari [**12**,16.1.i],we may write

$$E = E_1 \bigcup E_2 \tag{2.28}$$

a disjoint union of sets of equal measure such that

$$\int_{E_1} h_i d\lambda = \int_{E_2} h_i d\lambda, \; i = 1, ..., n \tag{2.29}$$

Define

$$u(t) = \begin{cases} 1 & \text{if } t \in E_1 \\ -1 & \text{if } t \in E_2 \\ 0 & \text{if } t \notin E_1 \bigcup E_2 \end{cases}$$

We shall show $w \pm \xi u \in W \bigcap G$ for ξ positive and sufficiently small. (2.29) yields that $w \pm \xi u \in G$ for $\xi > 0$. For $\xi \leq \frac{\epsilon}{3}$,we have $w \pm \xi u \geq 0$.

For $\sigma \notin [\tau - \epsilon, \tau + \epsilon]$ we have either $\sigma < \tau - \epsilon$ in which case

$$(w \pm \xi u - \sigma)_+ = (w - \sigma)_+ \pm \xi u \tag{2.30}$$

or we have $\sigma > \tau + \epsilon$ in which case

$$(w \pm \xi u - \sigma)_+ = (w - \sigma)_+. \tag{2.31}$$

In either case we obtain

$$\int_0^\infty (w \pm \xi u - \sigma)_+ d\lambda = \int_0^\infty (w - \sigma)_+ d\lambda. \tag{2.32}$$

If $\sigma \in [\tau - \epsilon, \tau + \epsilon]$ then

$$\begin{aligned} \int_0^\infty (w \pm \xi u - \sigma)_+ d\lambda &\leq \int_0^\infty (w - \sigma)_+ d\lambda + \int_0^\infty (\pm \xi u)_+ d\lambda && (2.33) \\ &< F(\sigma) + \xi \mu(E) \leq G(\sigma) - \delta + \xi \mu(E) && (2.34) \end{aligned}$$

where $\delta = \min\{G(s) - F(s) | s \in [\tau - \epsilon, \tau + \epsilon]\} > 0$, hence

$$\int_0^\infty (w \pm \xi u - \sigma)_+ d\lambda \leq G(\sigma) \tag{2.35}$$

for $0 < \xi \leq \min\{\frac{\epsilon}{3}, \frac{\delta}{\mu(E)}\}$.

Thus for small positive ξ ,

$$\int_0^\infty (w \pm \xi u - \sigma)_+ d\lambda \leq G(\sigma), \ \forall \sigma \in (0, \infty) \tag{2.36}$$

and hence $w \pm \xi u \in W \bigcap G$, so w is not extreme, a contradiction. We have shown that $w^{-1}(\Omega)$ has zero measure. We now consider three cases.

Case(i) (K, ∞) is a connected component of Ω. We will show $K = 0$ (and deduce that $w = 0$). Suppose $K > 0$, to seek a contradiction. Then we have

$$G(s) > F(s), \ s \in (K, \infty) \tag{2.37}$$

$$G(K) = F(K) \tag{2.38}$$

We have that $\mu\{w^{-1}(K, \infty)\} = 0$. It follows that $F(K + \epsilon) = 0, \forall \epsilon > 0$, and using continuity of F we have that $F(K) = 0$. But G is a monotone decreasing function, so for $s \in (K, \infty)$,

$$G(K) \geq G(s) > F(s) = 0 = F(K). \tag{2.39}$$

This contradicts $G(K) = F(K)$. Thus $K = 0$, whence $w = 0$.

Case(ii) Let (s_0, s_1) be a bounded connected component of Ω. We show $s_0 = 0$. Suppose, to seek a contradiction, that $s_0 > 0$. Then we have

$$F(s_1) = G(s_1) \tag{2.40}$$

$$F(s_0) = G(s_0) \tag{2.41}$$

$$G(s) > F(s) \text{ for } s \in (s_0, s_1) \tag{2.42}$$

Furthermore $\mu\{w^{-1}(s_0, s_1)\} = 0$. Then

$$F(s) = F(s_1) + \mu\{w^{-1}[s_1, \infty)\}(s_1 - s), \ \forall s \in [s_0, s_1]. \tag{2.43}$$

Therefore F is affine on (s_0, s_1) and

$$F(s_i) = G(s_i), \ i = 1, 2 \tag{2.44}$$

Because $F \leq G$, it follows by the convexity of G that $F = G$ on (s_0, s_1). This contradiction shows $s_0 = 0$.

Case(iii) We know that $\Omega = \emptyset, (0, K)$ or $(0, \infty)$ for some $K \in R$. If $\Omega = \emptyset$, then $w \in R(f_0) \subset RC(f_0)$. If $\Omega = (0, \infty)$, then $w^{-1}(0, \infty)$ has zero measure, whence $w = 0$, so $w \in RC(f_0)$. (w is a rearrangement of f_0 curtailed at 0).

It remains to consider the case $\Omega = (0, K)$. In this case we have

$$G(s) = F(s), s \in [K, \infty) \tag{2.45}$$

$$G(s) > F(s), s \in (0, K) \tag{2.46}$$

and $\mu\{w^{-1}(0, K)\} = 0$.

For $s \in (0, K)$,

$$\begin{aligned}
F(s) &= \int_0^\infty (w - s)_+ d\lambda \qquad (2.47)\\
&= \int_{w^{-1}(K,\infty)} (w - s)_+ d\lambda + \int_{w^{-1}(K)} (w - s)_+ d\lambda \\
&\quad + \int_{w^{-1}(0,K)} (w - s)_+ d\lambda + \int_{w^{-1}(0)} (w - s)_+ d\lambda \qquad (2.48)\\
&= F(K) + \mu\{w^{-1}(K, \infty)\}(K - s) + \mu\{w^{-1}(K)\}(K - s) \qquad (2.49)\\
&= F(K) + l(K - s) \qquad (2.50)
\end{aligned}$$

writing $l = \mu\{w^{-1}[K, \infty)\}$.

Now $F(s) = G(s)\ \forall s \in [K, \infty)$, so $w|_{w^{-1}(K,\infty)}$ is a rearrangement of $f_0|_{f_0^{-1}(K,\infty)}$. This implies $\mu\{f_0^{-1}(K,\infty)\} = \mu\{w^{-1}(K,\infty)\}$.

Suppose for a contradiction that $\mu\{w^{-1}[K,\infty)\} > \mu\{f_0^{-1}[K,\infty)\}$. The map Ψ, where $\Psi(s) = \mu\{f_0^{-1}[s,\infty)\}$, is left continuous. Thus for $s < K$, s sufficiently close to K,

$$\mu\{w^{-1}[K,\infty)\} > \mu\{f_0^{-1}[s,\infty)\} \tag{2.51}$$

Noting (2.51) and the fact that $w|_{w^{-1}(K,\infty)}$ and $f_0|_{f_0^{-1}(K,\infty)}$ are rearrangements, we see that $F(s) \geq G(s)$ for such s. This is a contradiction. Hence

$$\mu\{f_0^{-1}(K,\infty)\} = \mu\{w^{-1}(K,\infty)\} \leq l = \mu\{w^{-1}[K,\infty)\} \leq \mu\{f_0^{-1}[K,\infty)\} \tag{2.52}$$

Furthermore $w^{-1}(0,\infty)$ has measure l, $\mu\{w^{-1}(0,K)\} = 0$ and $w|_{w^{-1}(K,\infty)}$ and $f_0|_{f_0^{-1}(K,\infty)}$ are rearrangements. Combining the above, w is a rearrangement of f_0 curtailed at l, so $w \in RC(f_0)$.

This completes the proof.

Lemma 3 *Let f_0, p, and W be as in Lemma 1. Then $W = \overline{RC(f_0)}^w$.*

Proof Let $w \in W$. Let U be a weak neighbourhood of w. Then there exists $\epsilon > 0$ and $h_1, ..., h_n \in L^q(0,\infty)$ (where q is the conjugate exponent of p) such that

$$V = \{u \in L^p(0,\infty) | -\epsilon < \int_0^\infty wh_i d\lambda - \int_0^\infty uh_i d\lambda < \epsilon, i = 1, ...n\} \subset U \tag{2.53}$$

Let

$$G = \{u \in L^p(0,\infty) | \int_0^\infty h_i u d\lambda = \int_0^\infty wh_i d\lambda, i = 1, ..., n\} \subset U \tag{2.54}$$

By the Krein–Milman theorem, $W \bigcap G$ has an extreme point ($W \bigcap G$ is non–empty and weakly compact), so by the previous lemma yields $f_1 \in RC(f_0) \bigcap G$ such that $f_1 \in W \bigcap G \subset W \bigcap U$.

Therefore $U \bigcap RC(f_0) \neq \emptyset$, that is $W \subset \overline{RC(f_0)}^w$. But W is weakly closed and contains $RC(f_0)$. Thus $W = \overline{RC(f_0)}^w$. This completes the proof.

Lemma 4 *Let f_0, p, W and G be as in Lemma 2. Then*

$$W \bigcap G = \overline{convRC(f_0) \bigcap G} \tag{2.55}$$

Proof The Krein–Milman Theorem yields that, $W \bigcap G \subset \overline{\mathrm{conv}RC(f_0) \bigcap G}$, since by Lemma 2, $\mathrm{extr}W \bigcap G \subset RC(f_0) \bigcap G$. But $W \bigcap G$ is closed and convex, and contains $RC(f_0) \bigcap G$. Thus $\overline{\mathrm{conv}RC(f_0) \bigcap G} \subset W \bigcap G$. This completes the proof.

Lemma 5 *Let f_0, W and G be as in Lemma 2. Then $RC(f_0) \bigcap G \subset extrW \bigcap G$.*

Proof Let $w \in RC(f_0) \bigcap G$. Then $w \in W \bigcap G$. Moreover, w is a rearrangement of f_0 curtailed at l, for some $l \in \overline{\mathbf{R}}$. Let $w_1, w_2 \in W \bigcap G$ and let $0 < \alpha < 1$ be such that

$$(1-\alpha)w_1 + \alpha w_2 = w \tag{2.56}$$

We examine different values of l.

If $l = 0$, then $w = 0$. Since $w_1, w_2 \in W$, $w_1, w_2 \geq 0$. Thus by (2.56) $w_1 = w_2 = w = 0$. So w is extreme.

If $l = \infty$, then $w \in R(f_0)$. Let $\sigma > 0$ be given. Using (2.56) and the fact that w, w_1 and $w_2 \in W$ we have

$$\begin{aligned}
\int_0^\infty (f_0 - \sigma)_+ d\lambda &= \int_0^\infty (w - \sigma)_+ d\lambda && (2.57)\\
&= \int_0^\infty ((1-\alpha)w_1 + \alpha w_2 - \sigma)_+ d\lambda && (2.58)\\
&\leq (1-\alpha)\int_0^\infty (w_1 - \sigma)_+ d\lambda + \alpha \int_0^\infty (w_2 - \sigma)_+ d\lambda && (2.59)\\
&\leq \int_0^\infty (f_0 - \sigma)_+ d\lambda && (2.60)
\end{aligned}$$

From the above we see that all the inequalities are equalities, and further,

$$\int_0^\infty (w-\sigma)_+ d\lambda = \int_0^\infty (w_1-\sigma)_+ d\lambda = \int_0^\infty (w_2-\sigma)_+ d\lambda, \ \forall \sigma > 0 \tag{2.61}$$

Thus w, w_1 and w_2 are rearrangements of each other. Suppose (for a contradic tion) $w_1 \neq w_2$. w_1 and w_2 are rearrangements, so they are not positive multiples of each other. Now, by strict convexity of L^p, and (2.56),

$$\begin{aligned} ||w||_p &= ||(1-\alpha)w_1 + \alpha w_2||_p && (2.62)\\ &< (1-\alpha)||w_1||_p + \alpha||w_2||_p && (2.63)\\ &= ||w||_p. && (2.64) \end{aligned}$$

The latter equality follows because $w_1, w_2 \in R(w)$. This contradiction yields $w = w_1 = w_2$. This shows that w is extreme.

Now consider $0 < l < \infty$. Let K be chosen such that

$$\mu\{f_0^{-1}(K,\infty)\} \leq l \leq \mu\{f_0^{-1}[K,\infty)\} \tag{2.65}$$

For $\sigma \in [K,\infty)$, noting that $w_1, w_2 \in W$,

$$\begin{aligned} \int_0^\infty (f_0-\sigma)_+ d\lambda &= \int_0^\infty (w-\sigma)_+ d\lambda && (2.66)\\ &= \int_0^\infty ((1-\alpha)w_1 + \alpha w_2 - \sigma)_+ d\lambda && (2.67)\\ &\leq (1-\alpha)\int_0^\infty (w_1-\sigma)_+ d\lambda + \alpha\int_0^\infty (w_2-\sigma)_+ d\lambda && (2.68)\\ &\leq \int_0^\infty (f_0-\sigma)_+ d\lambda. && (2.69) \end{aligned}$$

From this calculation we see that all the inequalities are equalities, and further

$$\int_0^\infty (w-\sigma)_+ d\lambda = \int_0^\infty (w_1-\sigma)_+ d\lambda = \int_0^\infty (w_2-\sigma)_+ d\lambda, \ \forall \sigma \in [K,\infty). \tag{2.70}$$

From the above, $w^{|K}, w_1^{|K}$ and $w_2^{|K}$ are rearrangements of each other, where

$$w^{|K} = w|_{w^{-1}(K,\infty)}. \tag{2.71}$$

Now for $i = 1, 2$, using (2.56) we have

$$w_i^{-1}[K,\infty) \subset w_i^{-1}(0,\infty) \subset w^{-1}(0,\infty) \tag{2.72}$$

and that $w_i \in W$. Further

$$w_i^{-1}[K,\infty) \subset w_i^{-1}(0,\infty) \subset w^{-1}(0,\infty) \subset w^{-1}[K,\infty) \tag{2.73}$$

by the choice of K, possibly neglecting sets of zero measure.

Combining (2.70) and (2.73) we obtain

$$||w_i||_p \le ||w||_p,\ i = 1, 2 \tag{2.74}$$

Suppose $w_1 \neq w_2$. Since $w_1^{|K}$ and $w_2^{|K}$ are rearrangements, we know w_1 and w_2 are not positive multiples of each other. Then, by strict convexity of L^p with $||.||_p$, and inequality (2.74),

$$\begin{aligned} ||w||_p &= ||(1-\alpha)w_1 + \alpha w_2||_p & (2.75)\\ &< (1-\alpha)||w_1||_p + \alpha||w_2||_p & (2.76)\\ &\le ||w||_p. & (2.77) \end{aligned}$$

This is a contradiction, whence $w_1 = w_2 = w$. Thus w is extreme.

This completes the proof.

Lemma 6 *For f_0 as above, $\overline{convRC(f_0)} = \overline{convR(f_0)}$.*

Proof It suffices to show that $RC(f_0) \subset \overline{\mathrm{conv} R(f_0)}$. Using Mazur's Lemma, it is sufficient to show that for $\tau \in RC(f_0)$, there exists $\{\tau_n\}_{n=1}^{\infty} \subset R(f_0)$ such that $\tau_n \xrightarrow{w} \tau$.

Let $\tau \in RC(f_0)$. Then τ is a rearrangement of f_0 curtailed at l, which we denote f_l. At this juncture we restrict attention to showing that there exists $\{\Psi_n\}_{n=1}^{\infty} \subset R(f_0)$ such that $\Psi_n \xrightarrow{w} f_l$.

Let $g \in L^q(0,\infty)$, where q denotes the conjugate exponent of p. Define

$$\Psi_n(t) = \begin{cases} f_0^*(t) & \text{if } t \in [0,l] \\ 0 & \text{if } t \in (l, l+n] \\ f_0^*(t-n) & \text{if } t \in (l+n, \infty) \end{cases}$$

Then $\Psi_n \in R(f_0)$ $\forall n \in \mathbb{N}$. Now

$$\left| \int_0^{\infty} g(\Psi_n - f_l) d\lambda \right| = \left| \int_{l+n}^{\infty} g \Psi_n d\lambda \right| \leq ||f_0||_p ||g|_{(l+n,\infty)}||_q \tag{2.78}$$

using Holder's inequality. Further

$$||g|_{(l+n,\infty)}||_q \to 0 \text{ as } n \to \infty. \tag{2.79}$$

Thus $\Psi_n \xrightarrow{w} f_l$.

Let $E^+ = \{t | \tau(t) > 0\}$. Define $E = E^+ \bigcup (0,l)$. Then E has finite measure. (at most $2l$). By Ryff [**29**, propositions 1–3], there exists a measure preserving transformation $\phi : E \to E$ such that $\tau|_E^* \circ \phi = \tau|_E$. (In this case we do not define $\tau|_E$ outside E). It is immediate that $\tau|_E^* = f_l|_E$. Define $B : L^p(E) \to L^p(E)$ by $B(f) = f \circ \phi$. Then B is a bounded linear map, with $||B|| = 1$. Define

$$\tau_n(t) = \begin{cases} \Psi_n \circ \phi(t) & \text{if } t \in E \\ \Psi_n(t) & \text{if } t \notin E \end{cases}$$

$\tau_n \in R(f_0)$ $\forall n \in \mathbb{N}$, since ϕ is measure preserving.

Let $g \in L^q(0,\infty)$. Then

$$\left|\int_0^\infty (\tau - \tau_n) g d\lambda\right| \leq \left|\int_E ((f_l - \Psi_n)\circ \phi) g d\lambda\right| + \left|\int_{E^c} \Psi_n g d\lambda\right| \tag{2.80}$$

$$= \left|\int_E B(f_l - \Psi_n) g d\lambda\right| + \left|\int_{E^c} \Psi_n g d\lambda\right|. \tag{2.81}$$

Now $\Psi_n \xrightarrow{w} f_l$, so $B\Psi_n \xrightarrow{w} Bf_l$ since B is bounded.

We have, for $n \in \mathbb{N}$,

$$\left|\int_{E^c} \Psi_n g d\lambda\right| \leq \int_{E^c} \Psi_n |g| d\lambda \leq \int_{l+n}^\infty \Psi_n |g| d\lambda \leq ||f_0||_p || g|_{(l+n,\infty)} ||_q \tag{2.82}$$

using Holder's inequality.

Combining the above we have that

$$\left|\int_0^\infty (\tau - \tau_n) g d\lambda\right| \to 0 \text{ as } n \to \infty \tag{2.83}$$

that is, $\tau_n \xrightarrow{w} \tau$. This completes the proof.

Corollary 1 *Let f_0 be as above. Then $R(f_0)$ is weakly dense in $RC(f_0)$.*

Proof Let $\tau \in RC(f_0)$. By the proof of the previous lemma, there exists $\{\tau_n\}_{n=1}^\infty \subset R(f_0)$ such that $\tau_n \xrightarrow{w} \tau$. This completes the proof.

Theorem 1 *Let $f_0 \in L^p(0,\infty)$, $1 < p < \infty$, with f_0 non–negative. Let $h_1, ., h_n \in L^q(0,\infty)$ (where q denotes the conjugate exponent of p) and let $\alpha_1, ..., \alpha_n \in \mathbb{R}$. Define*

$$W = \{w \geq 0, w \text{ measurable} | \int_0^\infty (w - \sigma)_+ \leq \int_0^\infty (f_0 - \sigma)_+, \ \forall \sigma > 0\} \tag{2.84}$$

$$G = \{w \in L^p(0,\infty) | \int_0^\infty w h_i = \alpha_i, \ i = 1, ..., n\} \tag{2.85}$$

Then the following is true.

(i) W is a convex, weakly compact subset of $L^p(0,\infty)$.

(ii) $extr W \bigcap G = RC(f_0) \bigcap G$.

(iii) $\overline{conv RC(f_0) \bigcap G} = W \bigcap G$.

(iv) $W = \overline{R(f_0)}^w = \overline{conv R(f_0)}$.

Proof (i) Lemma 1.

(ii) Lemma 2 and Lemma 5.

(iii) Lemma 4.

(iv) Lemma 4 yields $W = \overline{\mathrm{conv} RC(f_0)}$. By Lemma 6 we have

$$W = \overline{\mathrm{conv} RC(f_0)} = \overline{\mathrm{conv} R(f_0)}. \tag{2.86}$$

Lemma 3 gives $W = \overline{RC(f_0)}^w$. Combining this with Corollary 1 , we obtain $W = \overline{R(f_0)}^w$. This completes the proof.

Lemma 7 *For non-negative non-zero $f_0 \in L^1(0,\infty)$, $\overline{R(f_0)}^w$ is contained in no weakly compact set.*

Proof $L^1(0,\infty)$ is a Banach space, therefore by the Eberlein–Smulian theorem [**14**, Theorem 1, page 430], weakly closed weakly sequentially compact sets are weakly compact, and vice versa.

Define, for $n \in \mathbb{N}$,

$$f_n(t) = \begin{cases} 0 & \text{if } t \in (0,n] \\ f_0(t-n) & \text{if } t \in (n,\infty) \end{cases}$$

Then $f_n \in R(f_0)$, $\forall n \in \mathbb{N}$.

Let $A = (0,a)$ for some finite $a > 0$. Now $1_A \in (L^1(0,\infty))^*$ (where this denotes the dual space of $L^1(0,\infty)$). Further,

$$\int_0^\infty f_n 1_A \to 0 \text{ as } n \to \infty. \tag{2.87}$$

Thus the only possible weak limit of $\{f_n\}_{n=1}^{\infty}$ is the zero function.

However $1_{(0,\infty)} \in (L^1(0,\infty))^*$ and

$$\int_0^\infty f_n 1_{(0,\infty)} = ||f_0||_1 \neq 0, \ \forall n \in \mathbf{N}. \tag{2.88}$$

Thus the sequence $\{f_n\}_{n=1}^{\infty}$ has no weak limit, and it is clear that no subsequence of $\{f_n\}_{n=1}^{\infty}$ has a weak limit. This completes the proof.

We state (without detailed proof) a lemma characterising the extreme points of a set of the form introduced in Lemma 2, for non–negative functions defined on the unit interval.

Lemma 8 *Let non–negative $f_0 \in L^p(0,1)$, for $1 < p < \infty$. Let μ denote 1–dimensional Lebesgue measure. Let $h_1, ..., h_n \in L^q(0,1)$ (where q denotes the conjugate exponent of p), and let $\alpha_1, ..., \alpha_n \in \mathbf{R}$. Define*

$$W = \left\{ w \geq 0 | \int_0^1 (w-\sigma)_+ d\mu \leq \int_0^1 (f_0 - \sigma)_+ d\mu, \ \forall \sigma > 0, \ ||w||_1 = ||f_0||_1 \right\} \tag{2.89}$$

$$G = \left\{ w \in L^p(0,1) | \int_0^1 w h_i d\mu = \alpha_i, \ i = 1, ..., n \right\} \tag{2.90}$$

Then extr $(W \bigcap G) = R(f_0) \bigcap G$.

We give an outline of the proof. For $w \in$ extr $(W \bigcap G)$, define

$$\Omega = \left\{ \sigma \in (0,1) | \int_0^1 (w-\sigma)_+ d\mu < \int_0^1 (f_0 - \sigma)_+ d\mu \right\}. \tag{2.91}$$

Using the methods of Lemma 2, and noting the constraint that $||w||_1 = ||f_0||_1$, we show that $\Omega = \emptyset$. This shows that extr $(W \bigcap G) \subset R(f_0) \bigcap G$. The reverse inclusion follows by arguments similar to those used in Lemma 5. The fact that $W = \overline{R(f_0)}^w$ now follows by the Krein–Milman Theorem.

Chapter 3

Extension to unbounded domains in $I\!R^n$

3.1 Introduction

In this chapter we extend the results of Chapter 2 to functions defined on an unbounded domain of $\mathbf{R}^n$, $n \in \mathbf{N}$. For a given function, defined on an unbounded domain of $\mathbf{R}^n$, we give a definition for the set of rearrangements of curtailments, extending the concept we introduced in Chapter 2. We prove the following theorem:

Theorem 2 Let non–negative $f_0 \in L^p(\Omega)$ for $1 < p < \infty$, where $\Omega \subset \mathbf{R}^n$ is open and unbounded, and of infinite measure. Define

$$W = \{w \geq 0, w \text{ measurable}| \int_\Omega (w-\sigma)_+ d\lambda \leq \int_\Omega (f_0 - \sigma)_+ d\lambda, \ \forall \sigma > 0\} \quad (3.1)$$

where λ is a non–zero, σ–finite, positive measure, absolutely continuous with respect to n–dimensional Lebesgue measure.

Then

(i) $\overline{R(f_0)}^w = \overline{\text{conv}R(f_0)} = W$.

(ii) $\mathrm{extr}W = RC(f_0)$.

(iii) W is weakly compact.

The proof of the above theorem requires some results concerning measure preserving transformations: these are recalled below.

3.2 Measure Preserving Transformations

We recall the concept of a measure preserving transformation. Let $(U, \mathcal{F}, \mu)$ and $(V, \mathcal{Y}, \nu)$ be measure spaces. Let $T : U \to V$ be a map.

Definition T is a *measurable transformation* if the inverse image of a measurable set is measurable.

T is *measurability preserving* if T is a bijection, with T, T^{-1} measurable transformations (as defined above).

T is a *measure preserving transformation* if T is measurability preserving, and $\mu T^{-1} = \nu$ (with respect to $\mathcal{Y}$).

A corollary to Halmos [**21**, Theorem C, page 163], yields the following result.

Theorem Let T be a measure preserving transformation between measure spaces $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{Y}, \nu)$. Let g be an extended real valued function on Y, then

$$\int_Y g d\nu = \int_X g \circ T d\mu \tag{3.2}$$

in the sense that if either integral exists, then so does the other and the two are equal.

We recall the definition of an outer regular measure.

Definition Let $(X, \mathcal{S}, \mu)$ be a measure space, where the σ–algebra $\mathcal{S}$ contains the Baire sets. (The Baire sets are the elements of the σ–ring generated by the compact G_δ sets: we remark that the Baire and Borel sets co–incide for $X \subset \mathbf{R}^n$).

μ is an *outer regular measure* if for every $E \in \mathcal{S}$ we have

$$\mu(E) = \inf\{\mu(G) | E \subset G, G \text{ open}, G \in \mathcal{S}\}. \tag{3.3}$$

The Theorem stated below plays an integral role in extending the work of Chapter 2 to unbounded domains in $\mathbf{R}^n$.

Theorem 1 *Let X be an open unbounded domain in $\mathbf{R}^n$, having infinite ν–measure, where ν is a non–zero σ–finite outer regular measure. Further, let $\nu\{x\} = 0$ for each set consisting of a single point $x \in X$. Let $(0, \infty)$ be endowed with Lebesgue measure. Then there exists a measure preserving transformation $\varphi : (0, \infty) \to X$.*

Proof Let $X = \bigcup_{n=1}^{\infty} X_n$, a disjoint union of ν–measurable sets such that $\nu(X_n) < \infty$, for each $n \in \mathbf{N}$. (Such a disjoint union exists since ν is σ–finite.) Fix $n \in \mathbf{N}$. ν is an outer regular measure, therefore we can find a sequence $\{U_m\}_{m=1}^{\infty}$ of open measurable sets such that $X_n \subset U_m$ and $\nu(U_m \backslash X_n) = \nu(U_m) - \nu(X_n) \leq \frac{1}{m}$, for each $m \in \mathbf{N}$. Writing $G_n = \bigcap_{m=1}^{\infty} U_m$ we have that $X_n \subset G_n$, and $\nu(X_n) = \nu(G_n)$. We write $G_n = X_n \bigcup S_n$, where S_n has zero ν–measure. G_n is a G_δ subset of $\mathbf{R}^n$ (since Ω is open). [**28**, Proposition 33, page 164] yields that (G_n, d_1) is a complete separable metric space where d_1 is a metric on G_n equivalent to the metric induced by Euclidean distance on G_n.

For $n \in \mathbf{N}$, define $Y_n = [\sum_{i=1}^{n-1} \nu(X_i), \sum_{i=1}^{n} \nu(X_i))$. [**27**, Theorem 9, page 270] gives the existence of a measure preserving transformation $\Psi_n : G_n \backslash L_n \to Y_n \backslash M_n$ where L_n is a set of zero ν–measure, and M_n is a set of Lebesgue measure zero. Now $X_n \backslash L_n \subset G_n \backslash L_n$, and $\Psi_n(X_n \backslash L_n) = Y_n \backslash M_n^{'}$, where $M_n^{'} = Y_n \backslash \Psi_n(X_n \backslash L_n)$. Now

$$\Psi_n : X_n \backslash L_n \to Y_n \backslash M_n^{'} \tag{3.4}$$

is a measure preserving transformation. L_n is a set of zero ν–measure. $M_n^{'}$ is a set of Lebesgue measure zero, because S_n has zero ν–measure and Ψ_n is a measure preserving transformation.

We later show that there exists $J_n \subset Y_n$, J_n a set of cardinal c (the cardinal number of the continuum) and Lebesgue measure zero, disjoint from M_n'. Now the sets $L_n \bigcup \Psi_n^{-1}(J_n)$ and $J_n \bigcup M_n'$ both have cardinal c, and $L_n \bigcup \Psi_n^{-1}(J_n)$ has ν–measure zero (using the fact that Ψ_n is a measure preserving transformation), while $J_n \bigcup M_n'$ has Lebesgue measure zero. Let $T_n : L_n \bigcup \Psi_n^{-1}(J_n) \to J_n \bigcup M_n'$ be a bijection (Each set has cardinal c therefore we can find such a T_n). Define $\varphi_n : X_n \to Y_n$ by

$$\varphi_n(x) = \begin{cases} \Psi_n(x) & \text{if } x \in X_n \backslash (L_n \bigcup \Psi_n^{-1}(J_n)) \\ T_n(x) & \text{if } x \in L_n \bigcup \Psi_n^{-1}(J_n) \end{cases}$$

Then φ_n is a measure preserving transformation. This holds for each $n \in \mathbf{N}$. Define $\varphi : X \to (0, \infty)$ by

$$\varphi|_{X_n} = \varphi_n. \tag{3.5}$$

Then φ is a measure preserving transformation, and φ^{-1} is a measure preserving transformation from $(0, \infty)$ to X.

It remains to show that there exists a set J_n of cardinal c, Lebesgue measure zero, J_n a subset of the interval $\left[\sum_{i=1}^{n-1} \nu(X_i), \sum_{i=1}^{n} \nu(X_i)\right)$, and disjoint from M_n'. This is equivalent to showing that for $S \subset [0,1]$, where $\mu(S) = 1$ (where μ denotes Lebesgue measure) that there exists $J \subset S$, where J is a set of Lebesgue measure zero and cardinal c. Now S contains a compact set K with $\mu(K) = a$, where $0 < a < 1$. Define, for $x \in [0,1]$, $f(x) = \mu([0,x] \bigcap K)$. Then f is continuous, $f(0) = 0$, $f(1) = a$. For $\alpha \in (0,a)$, define

$$\xi(\alpha) = \inf\{x | f(x) = \alpha\}. \tag{3.6}$$

The intermediate value theorem yields that the set $\{x | f(x) = \alpha\}$ is non–empty. We show that

(i) $\xi : (0,a) \to K$.

(ii) ξ is injective.

(iii) ξ is measure preserving for Lebesgue sets $A \subset (0,a)$.

Fix $0 < \alpha < a$. Then, for every $\epsilon > 0$ (such that $\xi(\alpha) - \epsilon \geq 0$) $f(\xi(\alpha) - \epsilon) < \alpha$ by the definition of $\xi(\alpha)$. Then, for ϵ as above, $K \bigcap (\xi(\alpha) - \epsilon, \xi(\alpha))$ has positive measure. Therefore $\xi(\alpha)$ is a limit point of K, whence $\xi(\alpha) \in K$ since K is compact. This shows (i).

Let $\xi(\alpha_1) = \xi(\alpha_2)$. Then by the continuity of f we have

$$\alpha_1 = f(\xi(\alpha_1)) = f(\xi(\alpha_2)) = \alpha_2. \tag{3.7}$$

This shows that ξ is injective.

To prove (iii), we first show that ξ is measure preserving for Borel sets $A \subset (0,a)$. ξ is an increasing, injective function, therefore ξ^{-1} is a Borel function, whence ξ maps Borel sets to Borel sets. We define

$$\Gamma = \{A \subset (0,a) | \mu(\xi(A)) = \mu(A)\}. \tag{3.8}$$

It is immediate that $\emptyset \in \Gamma$. Let $k \in K$. Suppose $[k - \epsilon, k] \bigcap K$ has positive measure for every $\epsilon > 0$. Then

$$\xi(f(k)) = \inf\{x | f(x) = f(k)\} = k. \tag{3.9}$$

where the last equality follows by our assumption. So $k \in \xi((0,a))$. Thus we obtain $\xi((0,a)) = K$ except possibly for a set of measure zero, so $(0,a) \in \Gamma$. Suppose $A \in \Gamma$. Now $\xi((0,a) \backslash A) = K \backslash \xi(A)$, except possibly for a set of measure zero, therefore

$$\mu\left(\xi((0,a) \backslash A)\right) \;=\; \mu(K \backslash \xi(A)) = a - \mu(\xi(A)) \tag{3.10}$$

$$= \quad a - \mu(A) = \mu\left((0,a)\backslash A\right). \tag{3.11}$$

Thus $(0,a)\backslash A \in \Gamma$. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of members of Γ. We show that $\bigcup_{n=1}^{\infty} A_n \in \Gamma$. Without loss of generality, we assume $\{A_n\}_{n=1}^{\infty}$ are disjoint. Then

$$\begin{aligned}
\mu\left(\xi(\bigcup_{n=1}^{\infty} A_n)\right) &= \mu\left(\bigcup_{n=1}^{\infty} \xi(A_n)\right) && (3.12)\\
&= \sum_{n=1}^{\infty} \mu\left(\xi(A_n)\right) && (3.13)\\
&= \sum_{n=1}^{\infty} \mu(A_n) = \mu(\bigcup_{n=1}^{\infty} A_n) && (3.14)
\end{aligned}$$

where (3.13) follows because the $\xi(A_n)$ are disjoint by the injectivity of ξ. Thus Γ is a σ–algebra.

We now show that Γ contains the open intervals which are subsets of $(0,a)$. Let $(a_1, b_1) \subset (0,a)$. We have

$$\begin{aligned}
b_1 - a_1 &= f\left(\xi(b_1)\right) - f\left(\xi(a_1)\right) && (3.15)\\
&= \mu\left[0, \xi(b_1)\right]\bigcap K\Big) - \mu\left([0, \xi(a_1)]\bigcap K\right) && (3.16)\\
&= \mu\left([\xi(a_1), \xi(b_1)]\bigcap K\right) && (3.17)\\
&= \mu\left(\xi(a_1, b_1)\right) && (3.18)
\end{aligned}$$

where we have used the fact that ξ is an increasing function. Γ contains the open intervals and is a σ–algebra, therefore it contains the Borel sets of $(0,a)$. We show that Γ contains the Lebesgue measurable subsets of $(0,a)$. A Lebesgue measurable subset E of $(0,a)$ is the disjoint union of a Borel set and a Lebesgue set of measure zero. We have previously seen that Γ contains the Borel sets. Every set of Lebesgue measure zero in $(0,a)$ is contained in a G_δ (in $(0,a)$) set of the same measure. Since Γ is a σ–algebra which contains the open subsets of $(0,a)$, it follows that all G_δ subsets of $(0,a)$ are contained in Γ. A subset of a

Borel set of measure zero is a Lebesgue set of measure zero. Therefore every set of zero Lebesgue measure in $(0, a)$ belongs to Γ . Thus E is the union of sets in Γ, therefore $E \in \Gamma$.

Let C denote the Cantor set on $(0, a)$ (without loss of generality we assume $a \in \mathbb{Q}$). C has cardinal c, and is of zero Lebesgue measure, and is a subset of $(0, a)$. Using the above results, we obtain that the set $\xi(C) = J$, say, has cardinal c, is of zero Lebesgue measure, and is a subset of K, whence a subset of S. This completes the proof.

In Theorem 1, we require that the non–zero measure η is

(i) outer regular.

(ii) σ–finite.

(iii) $\eta(\{x\}) = 0$ for each $x \in \Omega$.

For a non–zero σ–finite measure ν, absolutely continuous with respect to n–dimensional Lebesgue measure (which we denote μ), (ii) and (iii) are immediate. We show (i).

The Radon–Nikodym Theorem gives the existence of a measurable finite valued function f (the Radon–Nikodym derivative) such that

$$\nu(S) = \int_S f d\mu \tag{3.19}$$

for each measurable set S. Let E be a measurable set. If $\nu(E) = \infty$, there is nothing to prove. Otherwise, by the outer regularity of μ, there exists a sequence $\{G_n\}_{n=1}^{\infty}$ of measurable open supersets of E, such that $\mu(G_n) \to \mu(E)$ (note that we may choose the sequence such that $G_{n+1} \subset G_n$). Then $1_{G_n} f \to 1_E f$ μ almost everywhere. The Dominated Convergence Theorem yields that

$$\int_{G_n} f d\mu \to \int_E f d\mu \text{ as } n \to \infty \tag{3.20}$$

and appealing to (3.19), we have $\nu(G_n) \to \nu(E)$ as required. This shows (i). Thus we are able to apply Theorem 1 for non-zero σ-finite measures ν, absolutely continuous with respect to n-dimensional Lebesgue measure.

3.3 Definitions and Notation

Let $\Omega \subset \mathbb{R}^n$, Ω an unbounded domain, where Ω is a set of infinite measure. We endow Ω with a non-zero σ-finite measure λ absolutely continuous with respect to n-dimensional Lebesgue measure. Section 3.2 yields the existence of a measure preserving transformation $T : (0, \infty) \to \Omega$. Define a map $A : L^p(\Omega) \to L^p(0, \infty)$ by

$$A(f) = f \circ T. \tag{3.21}$$

A is well-defined, bounded and linear, with bounded inverse A^{-1}. We note that A is an isometry. Furthermore, for non-negative $f, g \in L^p(\Omega)$ we have

$$g \in R(f) \text{ if and only if } A(g) \in R(A(f)). \tag{3.22}$$

(These properties are shown in the proof of Theorem 2).

Definition Let non-negative $f_0 \in L^p(\Omega)$, $1 < p < \infty$. Let non-negative $g \in L^p(\Omega)$, for p as above. Then g is a *rearrangement of a curtailment of* f_0 if and only if *A(g)* is a rearrangement of a curtailment of $A(f_0)$. We write $RC(f_0)$ for the set of rearrangements of curtailments of f_0.

Using properties of A, and Chapter 2, we see that $g \in RC(f_0)$ if and only if one of the following is true:

(i) $g \in R(f_0)$.

(ii) $g \equiv 0$.

(iii) there exists $K \in R$ satisfying each of the following

$$\int_\Omega (g-\sigma)_+ = \int_\Omega (f_0-\sigma)_+, \ \forall\sigma \geq K \tag{3.23}$$

$$\lambda(g^{-1}(K)) \leq \lambda(f_0^{-1}(K)) \tag{3.24}$$

$$\lambda(g^{-1}(0,K)) = 0 \tag{3.25}$$

3.4 Rearrangements on unbounded domains of $I\!R^n$

Theorem 2 *Let non–negative $f_0 \in L^p(\Omega)$, $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ open, unbounded, and of infinite measure. Define*

$$W = \{w \geq 0, w \text{ measurable on } \Omega | \int_\Omega (w-\sigma)_+ d\lambda \leq \int_\Omega (f_0-\sigma)_+ d\lambda, \forall\sigma > 0\} \tag{3.26}$$

where λ is non–zero, σ–finite, and absolutely continuous with respect to n–dimen –sional Lebesgue measure.

Then

(I) $\overline{R(f_0)}^w = \overline{convR(f_0)} = W$.

(II) $extrW = RC(f_0)$.

(III) W is weakly compact.

Proof (I) Theorem 1 shows the existence of a measure preserving transformation $T : (0,\infty) \to \Omega$. Define $A : L^p(\Omega) \to L^p(0,\infty)$ by $A(f) = f \circ T$. It is immediate that A is linear. For $f \in L^p(\Omega)$,

$$||A(f)||_p = \left\{\int_0^\infty |f\circ T|^p d\mu\right\}^{\frac{1}{p}} = \left\{\int_0^\infty |f|^p \circ T d\mu\right\}^{\frac{1}{p}} \tag{3.27}$$

$$= \left\{\int_{\Omega} |f|^p d\lambda\right\}^{\frac{1}{p}} = ||f||_p \tag{3.28}$$

Thus A is well–defined, bounded and linear. The inverse $A^{-1} : L^p(0,\infty) \to L^p(\Omega)$, given by $A^{-1}(g) = g \circ T^{-1}$, is bounded and linear by a similar calculation to the above. Define

$$W^1 = \{w \geq 0 | \int_0^{\infty} (w-\sigma)_+ d\mu \leq \int_0^{\infty} ((f_0 \circ T) - \sigma)_+ d\mu, \ \forall \sigma > 0\} \tag{3.29}$$

(where μ denotes 1–dimensional Lebesgue measure).

We aim to show the following,

(i) $A(g) \in W^1$ if and only if $g \in W$.

(ii) $A(g) \in \overline{\text{conv}R(A(f_0))}$ if and only if $g \in \overline{\text{conv}R(f_0)}$.

(iii) $A(g) \in \overline{R(A(f_0))}^w$ if and only if $g \in \overline{R(f_0)}^w$.

If we can show the above, we appeal to the result of Theorem 1, Chapter 2, yielding $\overline{R(A(f_0))}^w = \overline{\text{conv}R(A(f_0))} = W^1$. This implies that

$$\overline{R(f_0)}^w = \overline{\text{conv}R(f_0)} = W. \tag{3.30}$$

To show (i), let $g \in W$. Now $g \geq 0$, therefore $g \circ T \geq 0$ (T is a measure preserving transformation). Moreover, for any $\sigma > 0$,

$$\int_{\Omega} (g-\sigma)_+ d\lambda \leq \int_{\Omega} (f_0 - \sigma)_+ d\lambda. \tag{3.31}$$

By Halmos [**21**, Theorem 2, page 163], it follows that

$$\int_0^{\infty} (g-\sigma)_+ \circ T d\mu \leq \int_0^{\infty} (f_0 - \sigma)_+ \circ T d\mu, \ \forall \sigma > 0. \tag{3.32}$$

Rewriting the above inequality,

$$\int_0^\infty (g \circ T - \sigma)_+ d\mu \leq \int_0^\infty (f \circ T - \sigma)_+ d\mu, \ \forall \sigma > 0 \tag{3.33}$$

whence $A(g) \in W^1$. The reverse implication follows by a similar argument.

To show (ii), let $g \in \overline{\text{conv}R(f_0)}$. Then there exists a sequence $\{g_n\}_{n=1}^\infty$ with $g_n \in \text{conv}R(f_0)$ $\forall n \in \mathbb{N}$, such that $||g_n - g||_p \to 0$. For each $n \in \mathbb{N}$, g_n is a convex combination of members of $R(f_0)$. It follows from the measure preserving properties of T that $h \in R(f_0)$ if and only if $A(h) \in R(A(f_0))$. A is a linear map. Therefore $A(g_n)$ is a convex combination of members of $R(A(f_0))$. Moreover

$$||A(g_n) - A(g)||_p = ||g_n - g||_p \to 0 \text{ as } n \to \infty. \tag{3.34}$$

Thus $A(g) \in \overline{\text{conv}R(A(f_0))}$. The reverse implication may be shown in an analogous manner.

To show (iii), suppose $g \notin \overline{R(f_0)}^w$. Then there exists U_1, a weak neighbourhood of g such that $U_1 \bigcap R(f_0) = \emptyset$. (We assume U_1 is weakly open without loss of generality). A is a homeomorphism from the weak topology of $L^p(\Omega)$ to the weak topology of $L^p(0,\infty)$. Thus $A(U_1)$ is a weakly open neighbourhood of $A(g)$. Suppose that $A(U_1) \bigcap R(A(f_0)) \neq \emptyset$. Then there exists $h \in A(U_1) \bigcap R(A(f_0))$. Further $A^{-1}h \in U_1 \bigcap R(f_0)$, which contradicts our original supposition. Thus $A(U_1) \bigcap R(A(f_0)) = \emptyset$, so $A(g) \notin \overline{R(A(f_0))}^w$.

For the reverse implication we may use an argument similar to the above, or simply note;

If $g \in \overline{R(f_0)}^w$, $g \in \overline{\text{conv}R(f_0)}$. Then by (ii), $A(g) \in \overline{\text{conv}R(A(f_0))}$. But Theorem 1, Chapter 2 yields

$$\overline{\text{conv}R(A(f_0))} = \overline{R(A(f_0))}^w. \tag{3.35}$$

Thus $A(g) \in \overline{R(A(f_0))}^w$. This completes the proof of (iii).

(II) We aim to prove the following

(iv) $A(g) \in \text{extr}W^1$ if and only if $g \in \text{extr}W$.

To prove (iv), let $A(g) \in \text{extr}W^1$. Let $g_1, g_2 \in W$, and $\lambda \in (0,1)$ be such that

$$g = (1-\lambda)g_1 + \lambda g_2. \tag{3.36}$$

A is a linear map, whence

$$A(g) = (1-\lambda)A(g_1) + \lambda A(g_2). \tag{3.37}$$

$A(g) \in \text{extr } W^1$, thus $A(g) = A(g_1) = A(g_2)$. Injectivity of A yields $g = g_1 = g_2$. Thus $g \in \text{extr}W$. The reverse implication follows by a similar argument.

By definition $A(g) \in RC(A(f_0))$ if and only if $g \in RC(f_0)$.

Theorem 1 of Chapter 2 states that $\text{extr}W^1 = RC(A(f_0))$, thus we may conclude $\text{extr}W = RC(f_0)$.

(III) A^{-1} is a homeomorphism from the weak topology of $L^p(0,\infty)$ to the weak topology of $L^p(\Omega)$. W^1 is weakly compact (Theorem 1, Chapter 2), thus $A^{-1}(W^1)$ is weakly compact. By the proof of part(I), $A^{-1}(W^1) = W$. Thus W is weakly compact. This completes the proof.

Chapter 4

Rearrangements of L^1 functions

In Burton [**7**, Theorem 6] and Burton and Ryan [**10**, Lemma 5] respectively, it was shown that for non–negative $v \in L^1(I)$, where I denotes the unit interval, we have

(i) $\overline{R(v)}^w = \overline{\text{conv}R(v)}$.

(ii) $\overline{R(v)}^w = \{w \geq 0 | \int_0^s w^* \leq \int_0^s v^*, \forall s, 0 < s < 1,\ ||w||_1 = ||v||_1\}$.

We cannot use the methods of Chapter 2 when seeking unbounded domain analogues of the above results. Lemma 7 of Chapter 2 yields that for non–negative, non–zero $f_0 \in L^1(0,\infty)$, $\overline{R(f_0)}^w$ is contained in no weakly compact set.

We prove the following theorems;

Theorem 1 For non–negative $f_0 \in L^1(0,\infty)$, $\overline{R(f_0)}^w$ is convex (therefore we have $\overline{\text{conv}R(f_0)} = \overline{R(f_0)}^w$).

Theorem 2 Let non–negative $f_0 \in L^1(\Omega,\nu)$, where $\Omega \subset \mathbf{R}^n$ is an unbounded set of infinite ν–measure, and ν is a non–zero, σ–finite measure absolutely continuous with respect to n–dimensional Lebesgue measure. Then

(i) $\overline{R(f_0)}^w$ is convex (whence $\overline{\text{conv}R(f_0)} = \overline{R(f_0)}^w$).

(ii) $\overline{R(f_0)}^w$ is contained in no weakly compact set unless $f_0 \equiv 0$.

Theorem 3 For f_0, Ω, ν as in Theorem 2, define

$$W = \{w \geq 0 | \int_\Omega (w - \sigma)_+ d\nu \leq \int_\Omega (f_0 - \sigma)_+ d\nu, \forall \sigma > 0, ||w||_1 = ||f_0||_1\}. \quad (4.1)$$

Then

(i) W is closed, bounded and convex.

(ii) $\overline{R(f_0)}^w \subset W$.

Theorem 1 follows using an approximation argument. This work is extended in Theorem 2 via the properties of measure preserving transformations discussed in Chapter 3. The last theorem follows by standard analysis. We have not been able to show that $\overline{R(f_0)}^w = W$.

Theorem 1 *Let f_0 be non-negative, $f_0 \in L^1(0, \infty)$. Then $\overline{R(f_0)}^w$ is convex, whence $\overline{conv R(f_0)} = \overline{R(f_0)}^w$.*

Proof (i) Let μ denote 1–dimensional Lebesgue measure. We later show that for $f_1, f_2 \in R(f_0)$, $0 < \lambda < 1$, we have $(1 - \lambda)f_1 + \lambda f_2 \in \overline{R(f_0)}^w$. Assuming this to be true, we now show $\overline{R(f_0)}^w$ is convex. Let $f_1, f_2 \in \overline{R(f_0)}^w$,and let $0 < \lambda < 1$. Let U be a convex weak neighbourhood of 0. Then, by the definition of weak closure

$$(f_1 + U) \bigcap R(f_0) \neq \emptyset \quad (4.2)$$

$$(f_2 + U) \bigcap R(f_0) \neq \emptyset \quad (4.3)$$

Let $g_1 \in (f_1 + U) \cap R(f_0)$, $g_2 \in (f_2 + U) \cap R(f_0)$. Then

$$\{(1 - \lambda)g_1 + \lambda g_2 + U\} \bigcap R(f_0) \neq \emptyset \quad (4.4)$$

by the above result. That is, there exists

$$h \in \{(1-\lambda)g_1 + \lambda g_2 + U\} \bigcap R(f_0). \tag{4.5}$$

Therefore

$$h \in \{(1-\lambda)f_1 + (1-\lambda)U + \lambda f_2 + \lambda U + U\} \bigcap R(f_0) \tag{4.6}$$

and using the convexity of U we have

$$h \in \{(1-\lambda)f_1 + \lambda f_2 + 2U\} \bigcap R(f_0). \tag{4.7}$$

Thus $(1-\lambda)f_1 + \lambda f_2 \in \overline{R(f_0)}^w$.

It remains to show that for $f_1, f_2 \in R(f_0)$, $\lambda \in (0,1)$ we have $(1-\lambda)f_1 + \lambda f_2 \in \overline{R(f_0)}^w$. Let non-negative $f \in L^1(0,\infty)$. For $m \in \mathbb{N}$, define

$$T_m f(x) = \begin{cases} f(\frac{x}{\lambda} + (1 - \frac{1}{\lambda})\frac{k-1}{2^m}) & \frac{k-1}{2^m} < x \le \frac{k-1+\lambda}{2^m} \\ 0 & \frac{k-1+\lambda}{2^m} < x \le \frac{k}{2^m} \end{cases}$$

for $k \in \mathbb{N}$.

$$S_m f(x) = \begin{cases} 0 & \frac{k-1}{2^m} < x \le \frac{k-1+\lambda}{2^m} \\ f(\frac{x}{(1-\lambda)} + (1 - \frac{1}{(1-\lambda)})\frac{k}{2^m}) & \frac{k-1+\lambda}{2^m} < x \le \frac{k}{2^m} \end{cases}$$

for $k \in \mathbb{N}$.

Then for each $m \in \mathbb{N}$, $T_m f$ is a rearrangement of $f(\lambda^{-1}x)$ and $S_m f$ is a rearrangement of $f((1-\lambda)^{-1}x)$. Thus $T_m f_1 + S_m f_2 \in R(f_0)$, each $m \in \mathbb{N}$. It remains to show that

$$T_m f_1 + S_m f_2 \xrightarrow{w} \lambda f_1 + (1-\lambda) f_2 \tag{4.8}$$

Let $\epsilon > 0$ be given. We can choose $N \in \mathbb{N}$ such that

$$\int_N^\infty f_1 d\mu < \epsilon \tag{4.9}$$

and

$$\int_N^\infty f_2 d\mu < \epsilon. \tag{4.10}$$

An immediate consequence is

$$\int_N^\infty T_m f_1 + S_m f_2 d\mu < \epsilon. \tag{4.11}$$

Firstly, we show that

$$T_m f_1 + S_m f_2|_{(0,N]} \to \lambda f_1 + (1-\lambda) f_2|_{(0,N]} \text{ as } m \to \infty \tag{4.12}$$

weakly in $L^1(0,N]$. Let V denote the set of step functions on $(0,N]$, with discontinuities only at dyadic rationals. Let $f, g \in V$. Let $y_1, ..., y_l$ be the points of discontinuity of g . Then $y_j = \frac{q_j}{2^{\beta_j}}$ for $1 \leq j \leq l$, where $\beta_j, q_j \in \mathbb{N}$. Let $\gamma = \max_{j=1,...,l}\{\beta_j\}$. We show that

$$\int_0^N T_m f g d\mu = \lambda \int_0^N f g d\mu \text{ for } m \geq \gamma \tag{4.13}$$

Let $m \geq \gamma$. Then g is constant on the intervals $(\frac{k-1}{2^m}, \frac{k}{2^m})$ for $k = 1, 2, ..., N2^m$. Therefore, for $1 \leq k \leq N2^m$

$$\int_{\frac{k-1}{2^m}}^{\frac{k}{2^m}} T_m f g d\mu = \int_{\frac{k-1}{2^m}}^{\frac{k-1+\lambda}{2^m}} f g d\mu = \lambda \int_{\frac{k-1}{2^m}}^{\frac{k}{2^m}} f g d\mu \tag{4.14}$$

Thus (4.13) holds.

Now let $f \in V$, and let $g \in L^\infty(0,N]$. There exists $g^\# \in V$ such that $||g - g^\#||_2 < \epsilon$. Then

$$\begin{aligned} & \left| \int_0^N T_m f g d\mu - \lambda \int_0^N f g d\mu \right| & (4.15) \\ \leq & \left| \int_0^N T_m f g^\# d\mu - \int_0^N T_m f g d\mu \right| + \left| \int_0^N T_m f g^\# d\mu - \lambda \int_0^N f g^\# d\mu \right| \end{aligned}$$

$$+\left|\lambda \int_0^N f g^{\#} d\mu - \lambda \int_0^N f g d\mu\right| \tag{4.16}$$

$$\leq ||T_m f||_2 ||g^{\#} - g||_2 + \left|\int_0^N T_m f g^{\#} d\mu - \lambda \int_0^N f g^{\#} d\mu\right|$$

$$+\lambda ||f||_2 ||g^{\#} - g||_2 \tag{4.17}$$

$$< \epsilon ||T_m f||_2 + \epsilon + \epsilon\lambda ||f||_2 \tag{4.18}$$

for $m \geq L(g^{\#}, \epsilon)$, say. (We have used the fact $f, g^{\#} \in V$ and (4.13)). Thus

$$\int_0^N T_m f g d\nu \to \lambda \int_0^N f g d\nu \text{ as } m \to \infty \tag{4.19}$$

for all $f \in V$, for all $g \in L^\infty(0, N]$.

Now let non–negative $f \in L^1(0, N]$. Let $g \in L^\infty(0, N]$. There exists $f^{\#} \in V$ such that $||f^{\#} - f||_1 < \epsilon$. Then

$$\left|\int_0^N T_m f g d\mu - \lambda \int_0^N f g d\mu\right| \tag{4.20}$$

$$\leq \left|\int_0^N T_m f g d\mu - \int_0^N T_m f^{\#} g d\mu\right| + \left|\int_0^N T_m f^{\#} g d\mu - \lambda \int_0^N f^{\#} g d\mu\right|$$

$$+\left|\lambda \int_0^N f^{\#} g d\mu - \lambda \int_0^N f g d\mu\right| \tag{4.21}$$

$$\leq ||T_m f - T_m f^{\#}||_1 ||g||_\infty + \left|\int_0^N T_m f^{\#} g d\mu - \lambda \int_0^N f^{\#} g d\mu\right|$$

$$+\lambda ||f^{\#} - f||_1 ||g||_\infty \tag{4.22}$$

$$< 2\lambda\epsilon ||g||_\infty + \epsilon \tag{4.23}$$

for $m \geq K(f^{\#}, \epsilon)$, noting that $f^{\#} \in V$ and using equation (4.19). Thus we have

$$T_m f|_{(0,N]} \xrightarrow{w} \lambda f|_{(0,N]} \tag{4.24}$$

for non-negative $f \in L^1(0,\infty)$. Similarly we can show that

$$S_m f|_{(0,N]} \xrightarrow{w} (1-\lambda) f|_{(0,N]} \tag{4.25}$$

for f as above. Combining the above results we see that

$$T_m f_1 + S_m f_2|_{(0,N]} \xrightarrow{w} \lambda f_1 + (1-\lambda) f_2|_{(0,N]} \text{ as } m \to \infty \tag{4.26}$$

Finally, for $g \in L^\infty(0,\infty)$,

$$\begin{aligned}
& \left| \int_0^\infty (T_m f_1 + S_m f_2) g d\mu - \int_0^\infty (\lambda f_1 + (1-\lambda) f_2) g d\mu \right| & (4.27) \\
\leq & \left| \int_0^N (T_m f_1 + S_m f_2) g d\mu - \int_0^N (\lambda f_1 + (1-\lambda) f_2) g d\mu \right| & \\
& + \left| \int_N^\infty (T_m f_1 + S_m f_2) g d\mu \right| + \left| \int_N^\infty (\lambda f_1 + (1-\lambda) f_2) g d\mu \right| & (4.28) \\
< & \ \epsilon + 2\epsilon ||g||_\infty & (4.29)
\end{aligned}$$

for m sufficiently large. This completes the proof.

Theorem 2 *Let non-negative $f_0 \in L^1(\Omega, \nu)$ where $\Omega \subset \mathbf{R}^n$, an unbounded set of infinite ν–measure, where the non-zero σ–finite measure ν is absolutely continuous with respect to n–dimensional Lebesgue measure. Then*

(i) $\overline{R(f_0)}^w$ is convex, whence $\overline{conv R(f_0)} = \overline{R(f_0)}^w$.

(ii) $\overline{R(f_0)}^w$ is contained in no weakly compact set unless $f_0 \equiv 0$.

Proof (i) Theorem 1 of Chapter 3 shows the existence of a measure preserving transformation T from the half–line with Lebesgue measure to Ω with measure ν. Define $A : L^1(\Omega) \to L^1(0,\infty)$ by

$$A(f) = f \circ T. \tag{4.30}$$

A is bounded and linear, with $||A|| = 1$. The inverse $A^{-1} : L^1(0,\infty) \to L^1(\Omega)$, given by $A^{-1}(g) = g \circ T^{-1}$ is also bounded and linear. A is a homeomorphism between the weak topologies of $L^1(\Omega)$ and $L^1(0,\infty)$. We show that

$$w \in \overline{R(f_0)}^w \text{ if and only if } A(w) \in \overline{R(A(f_0))}^w. \tag{4.31}$$

Let $w \in \overline{R(f_0)}^w$. Let U be a weak neighbourhood of $A(w)$. Then $A^{-1}(U)$ is a weak neighbourhood of w (A^{-1} is a homeomorphism between the weak topologies of $L^1(0,\infty)$ and $L^1(\Omega)$). Therefore

$$A^{-1}U \bigcap R(f_0) \neq \emptyset \tag{4.32}$$

whence

$$U \bigcap R(A(f_0)) \neq \emptyset. \tag{4.33}$$

Thus $A(w) \in \overline{R(A(f_0))}$. The reverse implication follows by a similar argument. This verifies (4.31).

Let $w_1, w_2 \in \overline{R(f_0)}^w$, and let $\lambda \in (0,1)$. Then by (4.31) $A(w_1), A(w_2) \in \overline{R(A(f_0))}^w$. Further, by convexity of $\overline{R(A(f_0))}^w$ (this was shown in Theorem 1), $\lambda A(w_1) + (1-\lambda)A(w_2) \in \overline{R(A(f_0))}^w$. By linearity of A, $A(\lambda w_1 + (1-\lambda)w_2) \in \overline{R(A(f_0))}^w$. Appealing to (4.31), we have $\lambda w_1 + (1-\lambda)w_2 \in \overline{R(f_0)}^w$. This completes the proof.

(ii) Suppose, for a contradiction, that $\overline{R(f_0)}^w$ is contained in a weakly compact set S. Then

$$\overline{R(A(f_0))}^w = A(\overline{R(f_0)}^w) \subset A(S) \tag{4.34}$$

where $A(S)$ is a weakly compact set since A is a homeomorphism between the weak topologies of $L^1(\Omega)$ and $L^1(0,\infty)$. This contradicts Lemma 7 of Chapter 2. This completes the proof.

Theorem 3 *For non-negative $f_0 \in L^1(\Omega, \nu)$, Ω, ν as in Theorem 2, we define*

$$W = \{w \geq 0 | \int_\Omega (w-\sigma)_+ d\nu \leq \int_\Omega (f_0 - \sigma)_+ d\nu, \ \forall \sigma > 0, \ ||w||_1 = ||f_0||_1\} \quad (4.35)$$

Then

(i) W is bounded, closed and convex.

(ii) $\overline{R(f_0)}^w \subset W$.

Proof (i) W is clearly bounded. For fixed $\sigma > 0$, we define the map $\Psi_\sigma : L^1(\Omega) \to \mathbb{R}$ by

$$\Psi_\sigma(f) = \int_\Omega (f-\sigma)_+ d\nu. \quad (4.36)$$

Ψ_σ is convex (immediate) and lower semicontinuous (follows by Fatou's Lemma).

Finally if $\{w_n\}_{n=1}^\infty \subset W$ and $w_n \to w$ we have

$$||w||_1 = ||\lim_{n\to\infty} w_n||_1 = \lim_{n\to\infty} ||w_n||_1 = ||f_0||_1 \quad (4.37)$$

and if $w_1, w_2 \in W$, $\lambda \in (0,1)$

$$\begin{aligned} ||(1-\lambda)w_1 + \lambda w_2||_1 &= \int_\Omega (1-\lambda) w_1 d\nu + \int_\Omega \lambda w_2 d\nu & (4.38) \\ &= (1-\lambda)\int_\Omega w_1 d\nu + \lambda \int_\Omega w_2 d\nu = ||f_0||_1. & (4.39) \end{aligned}$$

Combining the above results, W is convex and closed.

(ii) For $w \in R(f_0)$, for all $\sigma > 0$ we have

$$\int_\Omega (w-\sigma)_+ d\nu = \int_\Omega (f_0-\sigma)_+ d\nu. \quad (4.40)$$

Applying the Monotone Convergence Theorem, we obtain

$$\int_\Omega w d\nu = \lim_{\sigma\to 0} \int_\Omega (w-\sigma)_+ d\nu = \lim_{\sigma \to 0} \int_\Omega (f_0-\sigma)_+ d\nu = \int_\Omega f_0 d\nu \quad (4.41)$$

that is, $||w||_1 = ||f_0||_1$. Thus $w \in W$. W is a weakly closed set that contains $R(f_0)$, thus $\overline{R(f_0)}^w \subset W$. This completes the proof.

Chapter 5

An equivalent characterisation of the weak closure

5.1 Introduction

In Burton and Ryan [**10**, Lemma 5], it was shown that for non-negative $v \in L^p(I)$, where I denotes the unit interval and $1 \leq p < \infty$, that

$$\overline{R(v)}^w = \{w \geq 0 | \int_0^s w^* d\mu \leq \int_0^s v^* d\mu, \text{ for all } 0 < s < 1, ||w||_1 = ||v||_1\}. \quad (5.1)$$

In Chapters 2 and 4 we have seen that for non-negative $f_0 \in L^p(0, \infty)$, $1 \leq p < \infty$,

$$\{w \geq 0 | \textstyle\int_0^\infty (w - \sigma)_+ d\mu \leq \int_0^\infty (f_0 - \sigma)_+ d\mu, \forall \sigma > 0\} \begin{cases} = \overline{R(f_0)}^w & \text{if } 1 \leq p < \infty \\ \subset \overline{R(f_0)}^w & \text{if } p = 1 \end{cases}$$

We seek to show that the weak closure of the set of rearrangements can be characterised by a set of the form used in Burton and Ryan [**10**]. (The restriction that the 1–norms are equal must be discarded for L^p functions, $1 < p < \infty$, defined on the half–line: note the zero function is in the weak closure of the set of rearrangements.)

We prove the following theorem;

Theorem 1 For non–negative $f_0 \in L^p(0,\infty)$, $1 \leq p < \infty$, we define

$$W \quad = \{w \geq 0 | \int_0^\infty (w-\sigma)_+ d\mu \leq \int_0^\infty (f_0-\sigma)_+ d\mu, \forall \sigma > 0\} \tag{5.2}$$

$$W^1 \quad = \{w \geq 0 | \int_0^s w^* d\mu \leq \int_0^s f_0^* d\mu, \forall s > 0\} \tag{5.3}$$

Then $W = W^1$.

We prove Theorem 1 by showing that the functions, $\sigma \to \int_0^\infty (w-\sigma)_+ d\mu$ and $s \to \int_0^{-s} w^* d\mu$ have extensions to functions defined on the real line which are conjugate convex functions. The theory of such functions shows the equivalence of W and W^1. Note that our new characterisation of the weak closure of the set of rearrangements may be proved directly (if somewhat lengthily) by modifying the methods used by Burton and Ryan [**10**].

In Theorem 2 we prove an analogous result to the above, for non–negative L^p functions defined on the real line.

5.2 Proof of Theorem 1

Theorem 1 *Let be non–negative $f_0 \in L^p(0,\infty)$, for $1 \leq p < \infty$. Let μ denote 1–dimensional Lebesgue measure. Define*

$$W \quad = \{w \geq 0, w \ \textit{measurable} | \int_0^\infty (w-\sigma)_+ \leq \int_0^\infty (f_0-\sigma)_+, \ \forall \sigma > 0\} \tag{5.4}$$

$$W^1 \quad = \{w \geq 0, w \ \textit{measurable} | \int_0^s w^* d\mu \leq \int_0^s f_0^* d\mu, \ \forall s > 0\} \tag{5.5}$$

where h^* denotes the decreasing rearrangement of h on the half–line. Then

$$W^1 = W \begin{cases} = \overline{R(f_0)}^w & \textit{if } 1 < p < \infty \\ \supset \overline{R(f_0)}^w & \textit{if } p = 1 \end{cases}$$

where the relations in brackets follow by previous results.

Proof Note that it may be shown that $W^1 \subset L^p(0,\infty)$. Let non-negative $w \in L^p(0,\infty)$ for p as above. Define, for $\sigma \in \mathbf{R}$,

$$F(\sigma) = \begin{cases} \int_0^\infty (w-\sigma)_+ d\mu & \text{if } \sigma \geq 0 \\ \infty & \text{if } \sigma < 0 \end{cases}$$

$$F_0(\sigma) = \begin{cases} \int_0^\infty (f_0-\sigma)_+ d\mu & \text{if } \sigma \geq 0 \\ \infty & \text{if } \sigma < 0 \end{cases}$$

It is immediate that F and F_0 are non-negative functions. Let $\sigma_1, \sigma_2 \in \mathbf{R}$, and let $\lambda \in (0,1)$. If $(1-\lambda)\sigma_1 + \lambda\sigma_2 < 0$, then either $\sigma_1 < 0$ or $\sigma_2 < 0$. Therefore we have

$$F((1-\lambda)\sigma_1 + \lambda\sigma_2) = \infty = (1-\lambda)F(\sigma_1) + \lambda F(\sigma_2) \tag{5.6}$$

If $(1-\lambda)\sigma_1 + \lambda\sigma_2 = 0$, then either $\sigma_1 = \sigma_2 = 0$, else $\sigma_1 < 0$ or $\sigma_2 < 0$. The following formula then holds, with equality in the first case, and the right hand side being infinite in the second:

$$F((1-\lambda)\sigma_1 + \lambda\sigma_2) \leq (1-\lambda)F(\sigma_1) + \lambda F(\sigma_2). \tag{5.7}$$

If $(1-\lambda)\sigma_1 + \lambda\sigma_2 > 0$, then

$$\begin{aligned} F((1-\lambda)\sigma_1 + \lambda\sigma_2) &= \int_0^\infty (w - (1-\lambda)\sigma_1 - \lambda\sigma_2)_+ d\mu & (5.8) \\ &\leq (1-\lambda)\int_0^\infty (w-\sigma_1)_+ d\mu + \lambda\int_0^\infty (w-\sigma_2)_+ d\mu & (5.9) \\ &\leq (1-\lambda)F(\sigma_1) + \lambda F(\sigma_2). & (5.10) \end{aligned}$$

Thus F (and similarly F_0) is convex.

Let $\alpha \in \mathbf{R}$. Consider the set $\Gamma(\alpha) = \{\sigma | F(\sigma) \leq \alpha\}$ If $\alpha < 0$, $\Gamma(\alpha) = \emptyset$. For

$\alpha \geq 0$, $\alpha < \int_0^\infty w d\mu$,

$$\Gamma(\alpha) = \{\sigma | \int_0^\infty (w - \sigma)_+ d\mu \leq \alpha\} \tag{5.11}$$

which is a closed set because the function $\sigma \to \int_0^\infty (w - \sigma)_+ d\mu$ is continuous on $(0, \infty)$ (this was shown in the proof of Theorem 2, Chapter 2). If $\alpha \geq \int_0^\infty w d\mu$ (if this is possible) $\Gamma(\alpha) = [0, \infty)$, which is closed. Thus F (and similarly F_0) is lower semicontinuous.

Thus F and F_0 are non–negative, convex, lower semicontinuous functions. Ekeland and Temam [**15**, Proposition 3.1, page 14, page 17, and Proposition 4.1, page 18] yields that

$$F(\sigma) \leq F_0(\sigma), \forall \sigma \in \mathbf{R} \text{ if and only if } F^\#(s) \geq F_0^\#(s), \forall s \in \mathbf{R}. \tag{5.12}$$

where $H^\#$ denotes the conjugate function of H.

For $s \in \mathbf{R}$, we define

$$G(s) = \begin{cases} -\infty & \text{if } s < 0 \\ \int_0^s w^* d\mu & \text{if } s \geq 0 \end{cases}$$

$$G_0(s) = \begin{cases} -\infty & \text{if } s < 0 \\ \int_0^s f_0^* d\mu & \text{if } s \geq 0 \end{cases}$$

We show that $F^\#(s) = -G(-s)$ (respectively $F_0^\#(s) = -G_0(-s)$) for all $s \in \mathbf{R}$.

Let $s > 0$. $w \in L^p(0, \infty)$, therefore $F(\sigma) \to 0$ as $\sigma \to \infty$. Therefore we have

$$F^\#(s) = \sup_{\sigma \in \mathbf{R}} \{s\sigma - F(\sigma)\} = \infty = -G(-s). \tag{5.13}$$

Let $s = 0$. As noted above, $F(\sigma) \to 0$ as $\sigma \to \infty$, therefore

$$F^\#(0) = \sup_{\sigma \in \mathbf{R}} \{-F(\sigma)\} = 0 = -G(0). \tag{5.14}$$

We now consider $s < 0$. w^* is a rearrangement of w (by definition), therefore by the characterisation of the set of rearrangements given in Eydeland, Spruck and Turkington [**17**] we have

$$\int_0^\infty (w-\sigma)_+ d\mu = \int_0^\infty (w^*-\sigma)_+ d\mu \tag{5.15}$$

for all positive values of σ. We use this fact in the estimates below. For $\sigma > w^*(-s)$, we have

$$\begin{aligned} G(-s) - F(\sigma) &= \int_0^{-s} w^* d\mu - \int_0^\infty (w-\sigma)_+ d\mu && (5.16)\\ &= \int_0^{-s} w^* d\mu - \int_0^\infty (w^*-\sigma)_+ d\mu && (5.17)\\ &= \sigma\mu\{w>\sigma\} + \int_{\{x\in(0,-s)\mid w^*(x)\le\sigma\}} w^* d\mu && (5.18)\\ &\le \sigma\mu\{w>\sigma\} + \sigma(-s-\mu\{w>\sigma\}) = -\sigma s. && (5.19) \end{aligned}$$

For $0 < \sigma = w^*(-s)$,

$$\begin{aligned} G(-s) - F(\sigma) &= \int_0^{-s} w^* d\mu - \int_0^\infty (w-\sigma)_+ d\mu && (5.20)\\ &= \int_0^{-s} w^* d\mu - \int_0^\infty (w^*-\sigma)_+ d\mu = -\sigma s. && (5.21) \end{aligned}$$

For $0 < \sigma < w^*(-s)$, we have

$$\begin{aligned} G(-s) - F(\sigma) &= \int_0^{-s} w^* d\mu - \int_0^\infty (w-\sigma)_+ d\mu && (5.22)\\ &= \int_0^{-s} w^* d\mu - \int_0^\infty (w^*-\sigma)_+ d\mu && (5.23)\\ &= -\sigma s - \int_{-s}^\infty (w^*-\sigma)_+ d\mu \le -\sigma s. && (5.24) \end{aligned}$$

For $\sigma = 0$,

$$G(-s) - F(0) = \int_0^{-s} w^* d\mu - \int_0^{\infty} w^* d\mu \leq 0 = -s0. \tag{5.25}$$

Finally, for $\sigma < 0$,

$$G(-s) - F(\sigma) = \int_0^{-s} w^* d\mu - \infty = -\infty < -\sigma s. \tag{5.26}$$

Combining the above, we see that

$$G(-s) - F(\sigma) \leq -\sigma s, \ \forall \sigma \in \mathbf{R} \tag{5.27}$$

with equality holding in (5.27) for $\sigma = w^*(-s)$. Therefore

$$\sup_{\sigma \in \mathbf{R}} \{\sigma s - F(\sigma)\} = -G(-s) \tag{5.28}$$

whence,

$$F^{\#}(s) = -G(-s), \ \forall s \in \mathbf{R} \tag{5.29}$$

A similar argument shows that $F_0^{\#}(s) = -G_0(-s), \forall s \in \mathbf{R}$. Appealing to (5.12), we see

$$F(\sigma) \leq F_0(\sigma), \forall \sigma \in \mathbf{R} \text{ if and only if } G(s) \leq G_0(s), \forall s \in \mathbf{R}. \tag{5.30}$$

For $w \in W$, we have $F(\sigma) \leq F_0(\sigma), \forall \sigma \in \mathbf{R}$. For $w \in W^1$, we have $G(s) \leq G_0(s), \forall s \in \mathbf{R}$. Thus (5.30) yields that $W = W^1$. This completes the proof.

5.3 Symmetric decreasing rearrangements on $I\!R$

An analogue of Theorem 1 exists for non–negative functions defined on the real line. We define the (essentially unique) symmetric decreasing rearrangement of a non–negative function.

Definitions Let non–negative f be a Lebesgue measurable function defined on $\mathbf{R}$. We define the *distribution function* of f by

$$F(\alpha) = \mu(f^{-1}[\alpha, \infty)) \tag{5.31}$$

where μ denotes Lebesgue measure. The *symmetric decreasing rearrangement* of f, denoted $f^{\triangle}$, is defined by

$$f^{\triangle}(s) = \begin{cases} \max\{\alpha > 0 | F(\alpha) \geq 2|s|\} & \text{when this exists} \\ 0 & \text{otherwise} \end{cases}$$

Thus each set $f^{-1}[\alpha, \infty)$ is replaced by an interval of the same measure, symmetric about the origin.

Theorem 2 *Let non–negative $f_0 \in L^p(\mathbf{R})$ for $1 \leq p < \infty$. Let μ denote 1–dimensional Lebesgue measure. Define*

$$W = \{w \geq 0 | \int_{\mathbf{R}} (w - \sigma)_+ d\mu \leq \int_{\mathbf{R}} (f_0 - \sigma)_+ d\mu, \forall \sigma > 0\} \tag{5.32}$$

$$W^1 = \{w \geq 0 | \int_{-s}^{s} w^{\triangle} d\mu \leq \int_{-s}^{s} f_0^{\triangle} d\mu, \forall s > 0\} \tag{5.33}$$

where $h^{\triangle}$ denotes the symmetric decreasing rearrangement of h. Then

$$W^1 = W \begin{cases} = \overline{R(f_0)}^{w} & \textit{for } 1 < p < \infty \\ \supset \overline{R(f_0)}^{w} & \textit{for } p = 1 \end{cases}$$

where the relations in brackets follow by previous results.

Proof Let non–negative $w \in L^p(\mathbf{R})$ for p as above. Define, for $\sigma \in \mathbf{R}$,

$$F(\sigma) = \begin{cases} \int_{\mathbb{R}} (w-\sigma)_+ d\mu & \text{if } \sigma \geq 0 \\ \infty & \text{if } \sigma < 0 \end{cases}$$

$$F_0(\sigma) = \begin{cases} \int_{\mathbb{R}} (f_0-\sigma)_+ d\mu & \text{if } \sigma \geq 0 \\ \infty & \text{if } \sigma < 0 \end{cases}$$

For $s \in \mathbb{R}$, we define

$$G(s) = \begin{cases} -\infty & \text{if } s < 0 \\ \frac{1}{2}\int_{-s}^{s} w^{\triangle} d\mu & \text{if } s \geq 0 \end{cases}$$

$$G_0(s) = \begin{cases} -\infty & \text{if } s < 0 \\ \frac{1}{2}\int_{-s}^{s} f_0^{\triangle} d\mu & \text{if } s \geq 0 \end{cases}$$

The methods of the proof of Theorem 1 of this Chapter yield that

$$F(\sigma) \leq F_0(\sigma), \forall \sigma \in \mathbb{R} \text{ if and only if } G(s) \leq G_0(s), \forall s \in \mathbb{R}. \tag{5.34}$$

This completes the proof.

Chapter 6

Vortices in a Pipe and sets of Rearrangements

6.1 Introduction

We consider a variational problem, proposed by Benjamin [**3**], arising from a boundary value problem for an axisymmetric steady vortex ring in an ideal fluid flowing along an infinite pipe. The variational functional is shown to attain a maximum relative to the weak closure of the set of rearrangements of a fixed function, for all positive values of a parameter. The results obtained in Chapter 3 show that all maximisers belong to the set of rearrangements of curtailments of the original function.

We consider an ideal (inviscid and incompressible) fluid flowing without body forces. In view of the symmetry we work in a plane infinite strip, therefore we define

$$\Omega = \{(x, y) \in \mathbf{R}^2 | 0 < y < R\} \tag{6.1}$$

where $R > 0$ is fixed, and we endow Ω with the measure having Radon–Nikodym derivative $2\pi y$ with respect to plane Lebesgue measure.

Let h denote the fluid velocity field. The *vorticity* scalar field ξ is defined by

$$\text{curl } h = \xi \hat{k} \tag{6.2}$$

where $\hat{k}$ is a fixed unit vector perpendicular to the plane. In all axisymmetric (including unsteady) motions, the functions $\frac{\xi}{y}$ at any two instants are rearrangements of one another. An incompressible flow satisfies div $h = 0$ in Ω. Subject to suitable regularity assumptions, a stream function u exists, satisfying

$$h = \left(\frac{1}{y}u_y, -\frac{1}{y}u_x\right) \tag{6.3}$$

and using (6.2) we obtain

$$\frac{\xi}{y} = -\frac{1}{y}\left(\frac{1}{y}u_y\right)_y - \frac{1}{y^2}u_{xx}. \tag{6.4}$$

A differential operator $\mathcal{L}$ is defined on Ω by

$$\mathcal{L}u = -\frac{1}{y}\left(\frac{1}{y}u_y\right)_y - \frac{1}{y^2}u_{xx}. \tag{6.5}$$

We seek solutions of

$$\mathcal{L}u = \theta(u) \tag{6.6}$$

for some functions u and θ. (6.6) is the equation for the stream function of a steady flow. (A steady flow arises when there is some functional dependence between $\mathcal{L}u$ and u.) For a solution u to the problem to be studied, where u represents the stream function, the *vortex core* is the region where $\mathcal{L}u > 0$. At infinity the fluid velocity approaches a uniform stream of speed λ relative to the vortex core.

We define an inverse K for $\mathcal{L}$ by

$$\mathcal{L}Kv = v \text{ on } \Omega \tag{6.7}$$

and $Kv = 0$ on $\partial\Omega$ for $v \in L^p(\Omega)$, for suitable p (see Lemma 1). The boundary condition Kv is a constant is equivalent to requiring that the velocity field is tangential at the boundary. We take the constant to be zero. We can characterise Kv as the unique minimiser for $u \in H$ (where H is defined in Section 6.2) of the functional

$$\frac{1}{2}\int_\Omega \frac{1}{y^2}|\nabla u|^2 d\nu - \int_\Omega uv d\nu \tag{6.8}$$

where ν denotes the measure having density $2\pi y$ with respect to plane Lebesgue measure (see Lemma 1).

Benjamin's proposal was to seek extremals of the energy relative to an invariant set $R(f_0)$. For $\lambda > 0$ and $v \in L^p(\Omega)$ we consider the following variational functional

$$\Psi_\lambda(v) = \frac{1}{2}\int_\Omega vKv d\nu - \frac{1}{2}\lambda\int_\Omega vy^2 d\nu. \tag{6.9}$$

The former term of the functional represents kinetic energy and the latter momentum. It may be shown that

$$\frac{1}{2}\int_\Omega vKv d\mu = \frac{1}{2}\int_\Omega |h|^2 \, d\nu. \tag{6.10}$$

We prove the following theorem;

Theorem 1 Let f_0 be non–zero and non–negative, $f_0 \in L^1(\Omega)\bigcap L^p(\Omega)$, $1 < p < 2$. Then for any $\lambda > 0$,

(i) Ψ_λ attains a maximum value relative to $\overline{R(f_0)}^w$.

(ii) All maximisers of Ψ_λ relative to $\overline{R(f_0)}^w$ are members of $RC(f_0)$.

To show the above, we construct a maximising sequence for Ψ_λ relative to

$\overline{R(f_0)}^w$, with each function equal to its Steiner symmetrisation. (We recall this concept in Section 6.3.) $\overline{R(f_0)}^w$ is weakly sequentially compact, therefore the sequence has a subsequential weak limit (which is Steiner symmetric). The weak upper semicontinuity of Ψ_λ with respect to the Steiner symmetric elements of $\overline{R(f_0)}^w$ yields the existence of a maximiser. We have not been able to show that the maximiser is unique. Ψ_λ is a strictly convex function, therefore any maximiser is an extreme point of $\overline{R(f_0)}^w$. Theorem 2 of Chapter 3 yields that extr $\overline{R(f_0)}^w = RC(f_0)$, which shows Theorem 1 (ii).

Burton [**9**] considered the variational functional Ψ_λ with the constraint set the set of rearrangements of a fixed function with bounded support. It was shown that Ψ_λ attains a maximum relative to the set of rearrangements for all sufficiently small λ, whereas for λ tending to infinity, maximising sequences relative to the set of rearrangements were shown to tend weakly to 0 in $L^2(\Omega)$. Theorem 1 extends this work. Properties of the maximising functions are discussed in the next chapter.

The proof of the theorem stated above requires some results concerning Steiner symmetrisation. These are recalled in the section 6.3.

6.2 Definitions and Notation

In this section we write $L^p(\Omega, \eta)$ for the L^p space defined by a measure η. μ will denote plane Lebesgue measure. We seek a space appropriate to the study of K. This choice is guided by Burton [**9**, Section 3], who was in turn guided by Amick and Fraenkel [**2**, Section 2.2].

Definition Let H be the completion of the test functions on Ω with the scalar product

$$< u, v >_H = \int_\Omega y^{-2} \nabla u . \nabla v d\nu. \tag{6.11}$$

Ω is a strip, therefore the Hilbert space H is embedded in $H_0^1(\Omega, \mu)$, where plane Lebesgue measure is used to define $H_0^1(\Omega, \mu)$. By the Sobolev Embedding Theorem we have $H_0^1(\Omega, \mu)$ is embedded in $L^p(\Omega, \mu)$ for $2 \leq p < \infty$. Moreover $L^p(\Omega, \mu)$ is embedded in $L^p(\Omega, \nu)$ since the Radon–Nikodym derivative of ν with respect to μ is bounded above by a positive constant on Ω. Combining the above we obtain that H is embedded in $L^p(\Omega, \nu)$ for $2 \leq p < \infty$, for some embedding constant C.

Definition Let $(\Omega,\mathcal{Y},\nu)$ be a measure space, let $1 \leq p \leq \infty$, and let q be the conjugate exponent of p. A bounded linear operator $K : L^p(\Omega) \to L^q(\Omega)$ will be called *symmetric* if

$$\int_\Omega uKwd\nu = \int_\Omega wKud\nu \tag{6.12}$$

for all u and $w \in L^p(\Omega)$.

6.3 Steiner Symmetrisation

Let v be non–negative, $v \in L^p(\Omega, \nu)$, $1 \leq p < \infty$. We define the *Steiner symmetrisation* v^* of v with respect to the line $x = 0$ to be the essentially unique non–negative function in $L^p(\Omega, \nu)$ such that for each $\alpha > 0$ and almost every $y \in (0, R)$ the set

$$\{x | v^*(x, y) \geq \alpha\} \tag{6.13}$$

is an interval with centre 0, whose length equals the linear (1–dimensional Lebesgue) measure of the set

$$\{x | v(x, y) \geq \alpha\}. \tag{6.14}$$

For almost every $y \in (0, R)$, the Steiner symmetrisation v^* obeys

$$v^*(x, y) = v^*(-x, y) \tag{6.15}$$

$$v^*(x_1, y) \geq v^*(x_2, y) \tag{6.16}$$

for $0 \leq x_1 \leq x_2$, x, x_1 and $x_2 \in \mathbb{R}$. It is immediate that v^* is a rearrangement of v (with respect to ν).

We require some results on Steiner symmetrisation. Let non–negative $f, g \in L^p(\Omega, \nu)$, for p as above. For every $y \in (0, R)$ define

$$f_y(x) = f(x, y) \tag{6.17}$$

$$g_y(x) = g(x, y) \tag{6.18}$$

Crowe, Zweibel and Rosenbloom [**13**, Corollary 1], yields that

$$||f_y^\Delta - g_y^\Delta||_p \leq ||f_y - g_y||_p \text{ almost every } y \in (0, R) \tag{6.19}$$

where h^Δ denotes the symmetric decreasing rearrangement of h on $(-\infty, \infty)$.

By Fubini's theorem, we obtain

$$||f^* - g^*||_p \leq ||f - g||_p \tag{6.20}$$

where f^*, g^* denote the Steiner symmetrisation of f,g respectively, in the sense described above.

Let f be as above, and let non–negative $h \in L^q(\Omega, \nu)$, where q denotes the conjugate exponent of p. Then we have

$$\int_\Omega f h d\mu \leq \int_\Omega f^* h^* d\mu. \tag{6.21}$$

This inequality follows by similar reasoning to that used to obtain (6.20), using Crowe, Zweibel and Rosenbloom [**13**, Theorem 3], or alternatively Hardy, Littlewood and Polya [**22**, pages 260–299].

We state a result on Steiner symmetrisation from Appendix I of Fraenkel and Berger [**18**], where v^* was defined as above for non-negative continuous functions v with compact supports, and then defined for a general non-negative L^2 function by approximation in the 2-norm. It is easily verified that the two definitions are equivalent for non-negative $v \in L^2(\Omega, \nu)$. Fraenkel and Berger studied functions defined on a half-plane, but their results are equally applicable to functions on the strip Ω. For non-negative $u \in H$, (H defined as in Section 6.2) we have $u^* \in H$ and further

$$||u^*||_H \leq ||u||_H. \tag{6.22}$$

6.4 Existence of maximisers

Lemma 1 *Kv is well-defined for $v \in L^p(\Omega, \nu)$, where $1 < p \leq 2$. Further $K : L^p(\Omega, \nu) \to H$ is bounded and linear.*

Proof Let v be as above, and let q denote the conjugate exponent of p. We verify that Kv is the unique minimiser for $u \in H$ of the functional

$$\frac{1}{2}\int_\Omega \frac{1}{y^2}|\nabla u|^2 d\nu - \int_\Omega uv d\nu. \tag{6.23}$$

Applying Holder's inequality and the embedding discussed in Section 6.2 to the above functional for $u \in H$, we obtain

$$\begin{aligned} \frac{1}{2}||u||_H^2 - \int_\Omega uv d\nu &\geq \frac{1}{2}||u||_H^2 - ||v||_p||u||_q & (6.24)\\ &\geq \frac{1}{2}||u||_H^2 - C||v||_p||u||_H \to \infty \text{ as } ||u||_H \to \infty. & (6.25)\end{aligned}$$

Thus the functional is coercive on H. Further, it is strictly convex by the strict convexity of $||.||_H^2$, and it is weakly lower semicontinuous on H. Combining the above we obtain the existence of a unique minimiser, $u_0 \in H$ say, of the functional.

The Fréchet derivative of the functional on H at u_0 is identically zero. Therefore

$$\int_\Omega \frac{1}{y^2}\nabla u_0.\nabla h d\nu = \int_\Omega vhd\nu,\ \forall h \in H \tag{6.26}$$

whence

$$\int_\Omega \frac{1}{y^2}\nabla u_0.\nabla \phi d\nu = \int_\Omega v\phi d\nu,\ \forall \phi \in C_0^\infty(\Omega) \tag{6.27}$$

therefore

$$\int_\Omega u_0 \mathcal{L}\phi d\nu = \int_\Omega v\phi d\nu,\ \forall \phi \in C_0^\infty(\Omega). \tag{6.28}$$

Thus $\mathcal{L}u_0 = v$ in the weak sense, that is $Kv = u_0$.

By considering $u = 0$ for the functional (6.23), we obtain

$$\inf_{u\in H} \frac{1}{2}\int_\Omega \frac{1}{y^2}|\nabla u|^2 d\nu - \int_\Omega uvd\nu \leq 0. \tag{6.29}$$

Thus we have

$$\frac{1}{2}||u_0||_H^2 \leq \int_\Omega u_0 v d\nu \leq ||u_0||_q ||v||_p \leq C||u_0||_H ||v||_p \tag{6.30}$$

whence

$$||Kv||_H \leq C||v||_p. \tag{6.31}$$

Thus $K : L^p(\Omega,\nu) \to H$ is bounded. K is clearly linear. This completes the proof.

Lemma 2 $K : L^p(\Omega,\nu) \to H$ *is a positive (with respect to the usual order on function spaces) symmetric map, for p satisfying* $1 < p \leq 2$.

Proof Let non-negative $v \in L^p(\Omega,\nu)$, p as above. Then $Kv \geq 0$ by arguments we defer to the next chapter. (This result is shown in Lemma 2 of Chapter 7).

We show K is a symmetric operator. Let non-negative $v_1, v_2 \in L^p(\Omega,\nu)$, p as above. By Lemma 1, there exists $u_1, u_2 \in H$ such that $\mathcal{L}u_1 = v_1$ and $\mathcal{L}u_2 = v_2$ in the distributional sense. We show that

$$\int_\Omega u_1 \mathcal{L} u_2 d\nu = \int_\Omega \frac{1}{y^2} \nabla u_1 . \nabla u_2 d\nu \tag{6.32}$$

Let $\{\phi_n\}_{n=1}^{\infty} \subset C_0^{\infty}(\Omega)$ be a sequence of test functions such that $\phi_n \to u_1$ in H. By the Sobolev Embedding Theorem [**1**, Theorem 5.4B, page 97] we have $\phi_n \to u_1$ in $L^q(\Omega, \mu)$ where q denotes the conjugate exponent of p. Moreover, $\phi_n \to u_1$ in $L^q(\Omega, \nu)$. Note that $u_2 \in H$.

We have $\phi_n \to u_1$ in $L^q(\Omega, \nu)$ and $\phi_n \to u_1$ in H, therefore

$$\begin{aligned} \int_\Omega u_1 \mathcal{L} u_2 d\nu &= \lim_{n\to\infty} \int_\Omega \phi_n \mathcal{L} u_2 d\nu && (6.33) \\ &= \lim_{n\to\infty} \int_\Omega \frac{1}{y^2} \nabla \phi_n \nabla u_2 d\nu && (6.34) \\ &= \int_\Omega \frac{1}{y^2} \nabla u_1 \nabla u_2 d\nu && (6.35) \end{aligned}$$

This verifies (6.32).

Now by (6.32),

$$\int_\Omega u_1 \mathcal{L} u_2 d\nu = \int_\Omega u_2 \mathcal{L} u_1 d\nu, \tag{6.36}$$

Rewriting yields

$$\int_\Omega v_1 K v_2 d\nu = \int_\Omega v_2 K v_1 d\nu \tag{6.37}$$

whence K is a symmetric operator. This completes the proof.

Lemma 3 *For $\lambda > 0$, define*

$$\Psi_\lambda(\xi) = \frac{1}{2} \int_\Omega \xi K \xi d\nu - \frac{1}{2} \lambda \int_\Omega \xi y^2 d\nu \tag{6.38}$$

where non-negative $\xi \in L^p(\Omega, \nu)$, $1 < p \leq 2$.

Let ξ^* denote the Steiner Symmetrisation (with respect to the y-axis) of ξ.

Then

$$\Psi_\lambda(\xi^*) \geq \Psi_\lambda(\xi). \tag{6.39}$$

Proof For non-negative $w \in L^p(\Omega, \nu)$, p as above, we first show that

$$\int_\Omega w^* K w^* d\nu \geq \int_\Omega w K w d\nu. \tag{6.40}$$

From Lemma 1, we know $Kw \in H$. Using the methods of the proof of Lemma 2, we can show

$$\int_\Omega \frac{1}{y^2} \nabla Kw \nabla Kw d\nu = \int_\Omega Kw \mathcal{L} Kw d\nu. \tag{6.41}$$

Now

$$\begin{aligned} ||Kw||_H^2 &= \int_\Omega \frac{1}{y^2} |\nabla Kw|^2 d\nu && (6.42) \\ &= \int_\Omega Kw \mathcal{L} Kw d\nu && (6.43) \\ &= \int_\Omega w K w d\nu. && (6.44) \end{aligned}$$

We proceed using the methods of Burton [**9**, Lemma 2]. Kw is defined to be the unique minimiser over $h \in H$ of the functional

$$\frac{1}{2} \int_\Omega \frac{1}{y^2} |\nabla h|^2 d\nu - \int_\Omega h w d\nu. \tag{6.45}$$

Hence we have

$$-\frac{1}{2} \int_\Omega w K w d\nu = \inf_{h \in H} \{ \frac{1}{2} ||h||_H^2 - \int_\Omega h w d\nu \}. \tag{6.46}$$

Writing $Kw = u$, we have

$$-\frac{1}{2} \int_\Omega w K w d\nu = \frac{1}{2} ||u||_H^2 - \int_\Omega u w d\nu. \tag{6.47}$$

By methods analogous to those used to obtain (6.46), we have

$$-\frac{1}{2}\int_{\Omega} w^* K w^* d\nu = \inf_{h\in H}\{\frac{1}{2}||h||_H^2 - \int_{\Omega} hw^* d\nu\}. \tag{6.48}$$

From (6.48) we have

$$-\frac{1}{2}\int_{\Omega} w^* K w^* d\nu \leq \frac{1}{2}||u^*||_H^2 - \int_{\Omega} u^* w^* d\nu. \tag{6.49}$$

Now from (6.47) and (6.49),

$$\begin{aligned} & \frac{1}{2}\int_{\Omega} w^* K w^* d\nu - \frac{1}{2}\int_{\Omega} wKwd\nu & (6.50)\\ \geq\ & \int_{\Omega} u^* w^* d\nu - \frac{1}{2}||u^*||_H^2 + \frac{1}{2}||u||_H^2 - \int_{\Omega} uwd\nu \geq 0. & (6.51)\end{aligned}$$

using the Steiner symmetrisation inequalities (6.21) and (6.22).

Now for non-negative $w \in L^p(\Omega,\nu)$, wy^2 and w^*y^2 are ν-rearrangements. Thus

$$\int_{\Omega} wy^2 d\nu = \int_{\Omega} w^* y^2 d\nu \tag{6.52}$$

(in the sense that if either integral is finite, then so is the other, and they are equal. If we insist $w \in L^1(\Omega,\nu)\bigcap L^p(\Omega,\nu)$, the integrals are finite.)

Combining the above, for non-negative $\xi \in L^p(\Omega,\nu)$, $1 < p \leq 2$,

$$\begin{aligned}\Psi_\lambda(\xi^*) &= \frac{1}{2}\int_{\Omega} \xi^* K \xi^* d\nu - \frac{1}{2}\lambda\int_{\Omega} \xi^* y^2 d\nu & (6.53)\\ &\geq \frac{1}{2}\int_{\Omega} \xi K \xi d\nu - \frac{1}{2}\lambda\int_{\Omega} \xi y^2 d\nu = \Psi_\lambda(\xi). & (6.54)\end{aligned}$$

This completes the proof.

Corollary 1 *For* $\lambda > 0$, $\Psi_\lambda : L^1(\Omega,\nu)\bigcap L^p(\Omega,\nu) \to \mathbf{R}$ *is a strictly convex function, for* $1 < p \leq 2$.

Proof For $\xi \in L^1(\Omega,\nu) \bigcap L^p(\Omega,\nu)$, p as above,

$$\Psi_\lambda(\xi) = \frac{1}{2}\int_\Omega \xi K\xi d\nu - \frac{1}{2}\lambda\int_\Omega \xi y^2 d\nu. \tag{6.55}$$

Now

$$\int_\Omega \xi K\xi d\nu = ||K\xi||_H^2 \tag{6.56}$$

using the methods of the proof of Lemma 3. Thus $\xi \to \int_\Omega \xi K\xi d\nu$ is a strictly convex map by strict convexity of $||.||_H^2$ and linearity and injectivity of K.

The map $\xi \to \lambda\int_\Omega \xi y^2 d\nu$ is linear, whence Ψ_λ is strictly convex. This completes the proof.

Lemma 4 *Let $S(\Omega)_p^+$ denote the set of all non–negative Steiner symmetric functions in $L^p(\Omega,\nu)$. That is,*

$$S(\Omega)_p^+ = \{v \geq 0 | v \in L^p(\Omega,\nu), v \text{ symmetrically decreasing in } x \text{ a.e. } y \in (0,R)\} \tag{6.57}$$

Consider $T : S(\Omega)_p^+ \to \overline{\mathbf{R}}$, where

$$T(\xi) = \int_\Omega \xi K\xi d\nu \tag{6.58}$$

and $1 < p < 2$. Then T is weakly sequentially continuous.

Proof We show that

$$K(S(\Omega)_p^+) \subset \{w \geq 0,\ w \in H | w \text{ symmetrically decreasing in } x\}. \tag{6.59}$$

Applying the result of Lemma 1 we know that $K : L^p(\Omega,\nu) \to H$ is a bounded map. Using Lemma 2, we see that K is a positive map, that is $K\xi \geq 0$ for non–negative $\xi \in L^p(\Omega,\nu)$. It remains to show that for non–negative $\xi \in L^p(\Omega)$, p as above, ξ symmetrically decreasing in x, we have $K\xi$ symmetrically decreasing in x.

Let $\xi \in S(\Omega)_p^+$. $K\xi^*$ is defined to be the unique minimiser over $u \in H$ of the functional

$$\frac{1}{2}\int_\Omega \frac{1}{y^2}|\nabla u|^2 d\nu - \int_\Omega u\xi^* d\nu \equiv \Phi(u), \text{ say.} \tag{6.60}$$

Let $K\xi^* = u$. Then

$$\begin{aligned} \Phi(u^*) &= \frac{1}{2}\int_\Omega \frac{1}{y^2}|\nabla u^*|^2 d\nu - \int_\Omega u^*\xi^* d\nu && (6.61) \\ &\leq \frac{1}{2}\int_\Omega \frac{1}{y^2}|\nabla u|^2 d\nu - \int_\Omega u\xi^* d\nu && (6.62) \end{aligned}$$

using inequalities (6.21) and (6.22) from Section 6.3. Thus

$$\Phi(u^*) \leq \Phi(u). \tag{6.63}$$

Since u is the unique minimiser we have $u = u^*$, that is Kv^* is symmetrically decreasing in x.

By Lions [**24**, Theorem III.1] the set

$$\{u \geq 0 | u \in H_0^1(\Omega,\mu), u \text{ symmetrically decreasing in } x\} \tag{6.64}$$

is compactly embedded in $L^q(\Omega,\mu)$ where q denotes the conjugate exponent of p. (This result is valid for $2 < q < \infty$). We know from Section 6.2 that H is embedded in $H_0^1(\Omega,\mu)$ and further $L^q(\Omega,\mu)$ is embedded in $L^q(\Omega,\nu)$. Combining the above results, the map $K : S(\Omega)_p^+ \to L^q(\Omega,\nu)$ is compact.

We now show that T is weakly sequentially continuous. Let $u_n \xrightarrow{w} u_0$, where $\{u_n\}_{n=1}^\infty \subset S(\Omega)_p^+$, and $u_0 \in S(\Omega)_p^+$. Then

$$\left|\int_\Omega (u_0 - u_n)(Ku_0 - Ku_n) d\nu\right| \leq ||u_0 - u_n||_p ||Ku_0 - Ku_n||_q \tag{6.65}$$

and the right hand side tends to 0 as $n \to \infty$ because K is compact.

Thus

$$\lim_{n\to\infty} \int_\Omega (u_n - u_0)(Ku_n - Ku_0)d\nu = 0. \tag{6.66}$$

Rewriting we obtain

$$\lim_{n\to\infty} \left\{ \int_\Omega u_0 K u_0 d\nu + \int_\Omega u_n K u_n d\nu - \int_\Omega u_n K u_0 d\nu - \int_\Omega u_0 K u_n d\nu \right\} = 0 \tag{6.67}$$

whence

$$\int_\Omega u_0 K u_0 d\nu + \lim_{n\to\infty} \int_\Omega u_n K u_n d\nu - \int_\Omega u_0 K u_0 d\nu - \int_\Omega u_0 K u_0 d\nu = 0 \tag{6.68}$$

therefore

$$\lim_{n\to\infty} \int_\Omega u_n K u_n d\nu = \int_\Omega u_0 K u_0 d\nu. \tag{6.69}$$

Thus T is weakly sequentially continuous on $S(\Omega)_p^+$. This completes the proof.

Lemma 5 *Let non–negative $\xi \in L^p(\Omega, \nu)$, $1 \le p < \infty$. Then the map*

$$\xi \to \int_\Omega \xi y^2 d\nu \tag{6.70}$$

is weakly lower semicontinuous.

Proof Define $f : \Omega \times \mathbf{R} \to \overline{\mathbf{R}}$,

$$f((x,y),z) = \begin{cases} y^2 z & \text{if } z \ge 0 \\ \infty & \text{if } z < 0 \end{cases}$$

Then f is a non–negative normal integrand. Now let $\xi \in L^p(\Omega, \nu)$ for p as above. Define

$$F(\xi) = \int_\Omega f((x,y), \xi(x,y))d\nu. \tag{6.71}$$

Ekeland and Temam [**15**, Corollary 1.2, page 239] yields that F is a lower semicontinuous function. Furthermore since $f((x,y),.)$ is convex for almost every

$(x, y) \in \Omega$, F defines a convex function on $L^p(\Omega, \nu)$. Combining the above results, we obtain the fact that F defines a weakly lower semicontinuous function on $L^p(\Omega, \nu)$, for p as above.

It is elementary to show the restriction of F to non-negative $L^p(\Omega, \nu)$ functions is weakly lower semicontinuous. $F : (L^p(\Omega, \nu))^+ \to \overline{\mathbf{R}}$ may be written

$$F(u) = \int_\Omega uy^2 d\nu. \tag{6.72}$$

This completes the proof.

We state and prove a technical result we require in the proof of Theorem 1.

Lemma 6 *Let V be a vector space. Let $f : V \to \mathbf{R}$ be a strictly convex function, and w a maximiser for f relative to $U \subset V$. Then w is an extreme point of U.*

Proof We suppose the result is false, for a contradiction. Then there exists $w_1, w_2 \in U$, $w_1 \neq w_2$, and $\lambda \in (0, 1)$ such that

$$(1 - \lambda)w_1 + \lambda w_2 = w. \tag{6.73}$$

Strict convexity of f yields

$$\begin{aligned} f(w) &< (1-\lambda)f(w_1) + \lambda f(w_2) && (6.74)\\ &\leq (1-\lambda)f(w) + \lambda f(w) && (6.75)\\ &= f(w). && (6.76) \end{aligned}$$

This contradiction completes the proof.

Theorem 1 *Let non-negative, non-zero $f_0 \in L^1(\Omega, \nu) \bigcap L^p(\Omega, \nu)$, $1 < p < 2$. For every $\lambda > 0$,*

(i) Ψ_λ attains a maximum relative to $\overline{R(f_0)}^w$.

(ii) All maximisers in (i) belong to $RC(f_0)$.

Proof (i) Recall, for $\lambda > 0$, $\xi \in \overline{R(f_0)}^w$,

$$\Psi_\lambda(\xi) = \frac{1}{2}\int_\Omega \xi K \xi d\nu - \frac{1}{2}\lambda \int_\Omega \xi y^2 d\nu. \tag{6.77}$$

Let $\{\xi_n\}_{n=1}^\infty$ be a maximising sequence in $\overline{R(f_0)}^w$ for Ψ_λ. Replace ξ_n by ξ_n^*, the Steiner Symmetrisation of ξ_n with respect to the y-axis. Lemma 3 shows $\Psi_\lambda(\xi_n^*) \geq \Psi_\lambda(\xi_n)$ $\forall n \in \mathbb{N}$, thus $\{\xi_n^*\}_{n=1}^\infty$ is a maximising sequence in $\overline{R(f_0)}^w$ for Ψ_λ.

Let $\overline{R}^*$ denote the set of elements v of $\overline{R(f_0)}^w$ such that $v = v^*$. $\overline{R}^*$ is immediately seen to be convex, and the Steiner Symmetrisation inequality (6.20) in Section 6.3 yields that it is closed. Thus $\overline{R}^*$ is weakly closed. $\overline{R(f_0)}^w$ is weakly compact by Theorem 2 of Chapter 3, thus passing to a subsequence if necessary, $\xi_n^* \xrightarrow{w} \xi_0 \in \overline{R}^*$ (because $\overline{R}^*$ is weakly closed).

Appealing to Lemmas 4 and 5 we obtain

$$\begin{aligned}
\Psi_\lambda(\xi_0) &= \frac{1}{2}\int_\Omega \xi_0 K \xi_0 d\nu - \frac{1}{2}\lambda \int_\Omega \xi_0 y^2 d\nu && (6.78)\\
&\geq \lim_{n\to\infty} \frac{1}{2}\int_\Omega \xi_n^* K \xi_n^* d\nu - \frac{1}{2}\lambda \liminf_{n\to\infty} \int_\Omega \xi_n^* y^2 d\nu && (6.79)\\
&= \limsup_{n\to\infty} \frac{1}{2}\int_\Omega \xi_n^* K \xi_n^* d\nu + \limsup_{n\to\infty}\{-\frac{1}{2}\lambda \int_\Omega \xi_n^* y^2 d\nu\} && (6.80)\\
&\geq \limsup_{n\to\infty} \Psi_\lambda(\xi_n^*) \geq \sup_{\xi\in\overline{R(f_0)}^w} \Psi_\lambda(\xi). && (6.81)
\end{aligned}$$

Thus Ψ_λ attains a maximum relative to $\overline{R(f_0)}^w$.

(ii) For $\lambda > 0$, Ψ_λ is a strictly convex function. (This follows from Corollary 1). Let $\xi_0 \in \overline{R(f_0)}^w$ be a maximiser for Ψ_λ relative to $\overline{R(f_0)}^w$. Then ξ_0 is an extreme point of $\overline{R(f_0)}^w$, (by strict convexity of Ψ_λ and Lemma 6), therefore by Theorem 2 of Chapter 3 we have $\xi_0 \in RC(f_0)$. This completes the proof.

Chapter 7

Properties of the maximising functions

7.1 Introduction

We recall the variational problem we studied in Chapter 6. For $\lambda > 0$ and non-negative $w \in L^1(\Omega) \bigcap L^p(\Omega)$, $1 < p < 2$,

$$\Psi_\lambda(w) = \frac{1}{2}\int_\Omega wKw d\nu - \frac{1}{2}\lambda \int_\Omega wy^2 d\nu. \tag{7.1}$$

We recall from Chapter 6 that for non-negative $f_0 \in L^1(\Omega) \bigcap L^p(\Omega)$, ($p$ as above), Ψ_λ attains a maximum relative to $\overline{R(f_0)}^w$, and that the maximiser belongs to the rearrangements of the curtailments of f_0.

In Burton [**9**], Ψ_λ was shown to attain a maximum relative to the set of rearrangements of a fixed function with bounded support for all sufficiently small λ, whereas for all large λ, maximising sequences relative to the set of rearrangements were shown to tend weakly to 0 in $L^2(\Omega)$. For λ sufficiently large, maximisers relative to the set of rearrangements fail to exist. When a maximiser f_1 does exist, it was shown in Burton [**9**] that

$$f_1 = \phi(Kf_1 - \frac{1}{2}\lambda y^2) \tag{7.2}$$

for some increasing function ϕ, that is

$$\mathcal{L}u = \phi(u - \frac{1}{2}\lambda y^2) \tag{7.3}$$

where $Kf_1 = u$.

We show the following:

Theorem 3 Let non–negative $f_0 \in L^1(\Omega) \bigcap L^p(\Omega)$ for $p > \frac{5}{2}$. Fix $\lambda > 0$. Let $\tilde{f}$ be a maximiser for Ψ_λ relative to $\overline{R(f_0)}^w$. Then

(i) $\tilde{u} = K\tilde{f}$ satisfies

$$\mathcal{L}\tilde{u} = \phi(\tilde{u} - \frac{1}{2}\lambda y^2) \tag{7.4}$$

in the distributional sense, for some increasing function ϕ.

(ii) Except possibly for a set of measure zero,

$$\tilde{f}^{-1}(0,\infty) \subset (K\tilde{f} - \frac{1}{2}\lambda y^2)^{-1}(0,\infty). \tag{7.5}$$

Further $(K\tilde{f} - \frac{1}{2}\lambda y^2)^{-1}(0,\infty)$ is a bounded set, whence the set $\tilde{f}^{-1}(0,\infty)$ is bounded.

(iii) If $\tilde{f} \notin R(f_0)$,

$$(K\tilde{f} - \frac{1}{2}\lambda y^2)^{-1}(0,\infty) = \tilde{f}^{-1}(0,\infty) \tag{7.6}$$

except possibly for a set of measure zero, and $(K\tilde{f} - \frac{1}{2}\lambda y^2)^{-1}(0)$ has zero measure.

We show that the maximiser $\tilde{f}$ is the strict maximiser (relative to $R(f_0)$ or $\overline{R(f_0)}^w$ as appropriate) of a certain linear integral functional. We apply Theorem 1 of this chapter to obtain (i), and then use the fact that $\tilde{f}$ is a strict maximiser to obtain (ii) and (iii). We establish an upper bound for the size of the set

$(K\tilde{f}-\frac{1}{2}\lambda y^2)^{-1}(0,\infty)$ by using Newtonian potentials, and then applying maximum principles.

We apply the above theorem to obtain the following

Theorem 4 Let non-negative $f_0 \in L^1(\Omega)\bigcap L^p(\Omega)$ for $p > \frac{5}{2}$. Fix $\lambda > 0$. Let $\tilde{f}$ be a maximiser for Ψ_λ relative to $\overline{R(f_0)}^w$. Then there exists a sequence $\{\xi_n\}_{n=1}^\infty$ in $R(f_0)$, with weak limit $\tilde{f}$ in $L^p(\Omega)$, and $\Psi_\lambda(\xi_n) \to \Psi_\lambda(\tilde{f})$.

Thus, where $\tilde{f}$ is a maximiser for Ψ_λ (some fixed λ) relative to $\overline{R(f_0)}^w$, we can find a maximising sequence of rearrangements for Ψ_λ with weak limit $\tilde{f}$. The above theorem shows that $\sup_{f\in R(f_0)} \Psi_\lambda(f) = \sup_{f\in \overline{R(f_0)}^w} \Psi_\lambda(f)$. However we can find weakly convergent maximising sequences relative to $R(f_0)$ or $\overline{R(f_0)}^w$ where the weak limit is not a maximiser.

7.2 Equations satisfied by strict maximisers of functionals relative to sets of rearrangements

We begin by extending a theorem of Burton.

Theorem 1 *Let Ω be a domain in $\mathbf{R}^n$ with infinite ν–measure, where ν is absolutely continuous with respect to n–dimensional Lebesgue measure. Let non–negative $f_0 \in L^p(\Omega)$, for $1 \le p \le \infty$. Let $R(f_0)$ denote the set of rearrangements of f_0 on Ω. Let $g \in L^q(\Omega)$, where q denotes the conjugate exponent of p. Suppose that $\int_\Omega fgd\mu$ has a unique maximiser f^* relative to $R(f_0)$. Then there exists an increasing function ϕ such that $f^* = \phi \circ g$ ν–almost everywhere. (Note the possibility of ν being the zero measure is excluded).*

Proof We proceed using the methods of Burton [**7**, Theorem 2]. For $\alpha \in \mathbf{R}$ let

$$G_0(\alpha) = \{x \in \Omega | g(x) > \alpha\} \tag{7.7}$$

$$G(\alpha) = \{x \in \Omega | g(x) \geq \alpha\} \tag{7.8}$$

$$H(\alpha) = \{x \in \Omega | g(x) = \alpha\} \tag{7.9}$$

and define

$$\phi(\alpha) = \text{essinf } f^*(G(\alpha)). \tag{7.10}$$

It is immediate that $\phi : \mathbf{R} \to \overline{\mathbf{R}}$ is a non–decreasing function.

We show that $f^*(x) \leq \phi(\alpha)$ almost every $x \in \Omega \backslash G_0(\alpha)$. We suppose not, for a contradiction. Then there exists $\beta \in \mathbf{R}$ and $B^1 \subset \Omega \backslash G_0(\alpha)$, a set of finite positive measure (we can choose B^1 to have finite measure because ν is σ–finite) such that

$$f^*(x) > \beta > \phi(\alpha), \ \forall x \in B^1. \tag{7.11}$$

By the definition of ϕ, there exists $C^1 \subset G(\alpha)$, a set of finite positive measure such that

$$\phi(\alpha) \leq f^*(x) < \beta, \ \forall x \in C^1. \tag{7.12}$$

We may choose measurable sets $B \subset B^1, C \subset C^1$, such that B and C have equal positive finite measure. To see this, without loss of generality, suppose $0 < \nu(B^1) < \nu(C^1)$, and let $x \in \Omega$. Define $\eta : \mathbf{R} \to \mathbf{R}$ by

$$\eta(s) = \nu(B_s(x) \bigcap C^1) \tag{7.13}$$

where $B_s(x)$ denotes the open ball of radius s centred at $x \in \Omega$. Let f denote the Radon–Nikodym derivative of ν with respect to n–dimensional Lebesgue measure, which we denote μ. Then

$$\eta(s) = \int_{B_s(x)\bigcap C^1} f d\mu \tag{7.14}$$
$$= \int_{\Omega} 1_{B_s(x)\bigcap C^1} f d\mu. \tag{7.15}$$

Let $s_n \to s \in \mathbf{R}$. Then $1_{B_{s_n}(x)\bigcap C^1} f \to 1_{B_s(x)\bigcap C^1} f$ pointwise in Ω, and further

$$|1_{B_{s_n}(x)\bigcap C^1}(z)f(z)| \leq |1_{C^1}(z)f(z)| \text{ for } z \in \Omega. \tag{7.16}$$

Applying the Dominated Convergence Theorem, we obtain $\eta(s_n) \to \eta(s)$, so η is a continuous function. Accordingly we apply the intermediate value theorem to obtain the existence of $t \in \mathbf{R}$ such that $\eta(t) = \nu(B^1)$. Choose $C = C^1 \bigcap B_t(x)$, $B = B^1$. Let $\sigma : B \to C$ be a measure preserving transformation. Define $T : L^p(\Omega) \to L^p(\Omega)$ as follows

$$Tf(x) = \begin{cases} f(\sigma^{-1}(x)) & x \in C \\ f(\sigma(x)) & x \in B \\ f(x) & x \in \Omega \backslash B \bigcup C \end{cases}$$

(Clearly $B \bigcap C = \emptyset$, so T is well defined.) Now $Tf \in R(f)$ (because σ is a measure–preserving transformation). We aim to show that

$$\int_{\Omega} Tf^* g d\nu \geq \int_{\Omega} f^* g d\nu. \tag{7.17}$$

Now

$$\int_{\Omega}(Tf^* - f^*)gd\nu = \int_{B\bigcup C}(Tf^* - f^*)gd\nu = \int_{\Omega}(Tf^* - f^*)(g - \alpha)d\nu \tag{7.18}$$

because $\int_B Tf^* d\nu = \int_C f^* d\nu$, and $\int_B f^* d\nu = \int_C Tf^* d\nu$. The above integrals are finite, because $f \in L^p(B)$ implies $f \in L^1(B)$, since B has finite measure. Further $f \in L^p(B)$ implies $Tf \in L^p(C)$ because σ is a measure preserving transformation.

Now

$$\int_B Tf^*(g-\alpha)d\nu \geq \int_B \beta(g-\alpha)d\nu \text{ since } Tf^* < \beta, g \leq \alpha \text{ on } B. \tag{7.19}$$

$$\int_B f^*(g-\alpha)d\nu \leq \int_B \beta(g-\alpha)d\nu \text{ since } f^* > \beta, g \leq \alpha \text{ on } B. \tag{7.20}$$

$$\int_C Tf^*(g-\alpha)d\nu \geq \beta \int_C (g-\alpha)d\nu \text{ since } Tf^* > \beta, g \geq \alpha \text{ on } C. \tag{7.21}$$

$$\int_C f^*(g-\alpha)d\nu \leq \beta \int_C (g-\alpha)d\nu \text{ since } f^* < \beta, g \geq \alpha \text{ on } C. \tag{7.22}$$

Combining (7.19)–(7.22) we obtain

$$\int_\Omega (Tf^* - f^*)g d\nu \geq 0. \tag{7.23}$$

This contradicts the uniqueness of the maximiser f^*. Thus we have $f^*(x) \leq \phi(\alpha)$ almost every $x \in \Omega \backslash G_0(\alpha)$. In particular, this implies $f^*(x) \leq \phi(\alpha)$ for almost every $x \in H(\alpha)$. However $H(\alpha) \subset G(\alpha)$, thus by the definition of ϕ we have $f^*(x) \geq \phi(\alpha)$ for almost every $x \in H(\alpha)$, whence

$$f^*(x) = \phi(\alpha) \text{ for almost every } x \in H(\alpha). \tag{7.24}$$

We now show that $f^*(x) = \phi \circ g(x)$ almost every $x \in \Omega$. Suppose not, for a contradiction. Then there exists $\beta \in \mathbf{R}$, and S a set of positive measure such that either

$$f^*(x) < \beta < \phi \circ g(x), \ \forall x \in S. \tag{7.25}$$

or

$$f^*(x) > \beta > \phi \circ g(x), \ \forall x \in S. \tag{7.26}$$

In either case by (7.24), we have

$$\nu\{H(\alpha)\bigcap S\} = 0 \text{ for all } \alpha \in \mathbf{R}. \tag{7.27}$$

We show that there exists $\alpha \in \mathbf{R}$ such that both $S\bigcap G(\alpha)$ and $S\backslash G(\alpha)$ have positive measure. We suppose not. Let $\gamma = \sup\{t|\nu(S\bigcap G(t)) = \nu(S)\}$. It follows that $\nu(H(\gamma)\bigcap S) = \nu(S) > 0$. This contradicts (7.24).

Suppose (7.25) holds. The definition of $G(\alpha)$ yields that for every $x \in S\backslash G(\alpha)$, $g(x) < \alpha$. ϕ is a non–decreasing function, thus $\forall x \in S\backslash G(\alpha)$, $\phi(g(x)) \le \phi(\alpha)$. By (7.25) we have $\beta < \phi(\alpha)$. For almost every $x \in S\bigcap G(\alpha)$, $f^*(x) \ge \phi(\alpha)$. By (7.25) we have $\phi(\alpha) < \beta$. This is a contradiction.

Suppose that (7.26) holds. The definition of $G(\alpha)$ yields that $\forall x \in S\bigcap G(\alpha)$, $g(x) \ge \alpha$. ϕ is a non–decreasing function, thus $\phi(g(x)) \ge \phi(\alpha)$. By (7.26) $\beta > \phi(\alpha)$. For almost every $x \in S\backslash G(\alpha)$, $f^*(x) \le \phi(\alpha)$ (using our earlier result). By (7.26), $\beta < \phi(\alpha)$. This is a contradiction.

Thus $f^*(x) = \phi \circ g(x)$ for almost every $x \in \Omega$. This completes the proof.

7.3 Spaces appropriate to the study of K and $\mathcal{L}$

In our choice of spaces to study $\mathcal{L}$ we have been guided by Burton [**9**, Section 3.1] who was in turn guided by Amick and Fraenkel [**2**, Section 2.2].

Let U be the cylinder in $\mathbf{R}^5$ comprising all points whose distances from the x–axis are less than R. We regard Ω as the intersection of U with a half–plane bounded by the x–axis. Let y denote distance from the x–axis. Cylindrical symmetry in $\mathbf{R}^5$ is understood to be relative to the x–axis. We identify functions defined almost everywhere on Ω with cylindrically symmetric functions defined almost everywhere on U. With this identification we have

$$\mathcal{L}(y^2 u) = -\triangle_5 u \tag{7.28}$$

for smooth functions u defined on Ω, where $\triangle_5$ is the 5–dimensional Laplacian.

Let H be defined as in Section 6.2. Then, for non–negative non–zero $v \in L^p(\Omega)$, $1 < p \leq 2$, the operator $K : L^p(\Omega) \to L^q(\Omega)$ (where q denotes the conjugate exponent of p) satisfies the following

$$\mathcal{L}Kv = v \text{ on } \Omega \tag{7.29}$$

$$Kv = 0 \text{ on } \partial\Omega \tag{7.30}$$

where Kv may be characterised as the unique minimiser over $u \in H$ of the convex functional

$$\Psi_H^v(u) = \frac{1}{2}||u||_H^2 - \int_\Omega uv d\nu. \tag{7.31}$$

It has been shown previously that K is positive and a symmetric operator.

Define E to be the completion of the test functions on U (by which we mean $C_0^\infty(U)$ functions) with the scalar product

$$< u, v >_E = \int_U \nabla u . \nabla v d\tau \tag{7.32}$$

where the measure τ has Radon–Nikodym derivative (or density) $\frac{1}{\pi}$ with respect to 5–dimensional Lebesgue measure. U is a cylinder, thus it follows by Poincaré's inequality that E is a renorming of $W_0^{1,2}(U)$. For non–negative, non–zero $w \in L^p(U)$, $\frac{10}{7} \leq p \leq 2$, there exists unique $\mathcal{K}w \in E$ such that $-\triangle_5 u = w$ in the weak sense. We may characterise $\mathcal{K}w$ as the unique minimiser over $u \in E$ of

$$\Psi_E^w(u) = \frac{1}{2}||u||_E^2 - \int_U uw d\tau. \tag{7.33}$$

Then $\mathcal{K} : L^p(U) \to L^q(U)$ (where q denotes the conjugate exponent of p) is a positive and symmetric operator. (This follows by similar arguments to those used previously).

Lemma 1 *Let non-negative, non-zero $v \in L^1(\Omega)\bigcap L^p(\Omega)$, where $\frac{10}{7} \leq p \leq 2$. Then $\mathcal{K}v = y^{-2}Kv$.*

Proof For completeness we repeat the arguments of Burton [**9**, Lemma 1]. We show that $y^{-2}Kv$ satisfies

$$< y^{-2}Kv, h >_E - \int_U vh d\tau = 0 \tag{7.34}$$

for every $h \in E$. $\mathcal{K}v$ is the unique critical point of Ψ_E^v, thus we obtain $y^{-2}Kv = \mathcal{K}v$.

Firstly, we show that $y^{-2}Kv \in E$. Let $u, v \in C_0^\infty(\Omega)$. Then, integrating by parts,

$$\int_\Omega \frac{1}{y^2}\nabla u.\nabla v d\nu = \int_\Omega u\mathcal{L}v d\nu \tag{7.35}$$

$$= \int_U y^{-2}u(-\triangle y^{-2}v)d\tau = \int_U \nabla y^{-2}u\nabla y^{-2}v d\tau \tag{7.36}$$

That is,

$$< u, v >_H = < y^{-2}u, y^{-2}v >_E . \tag{7.37}$$

Therefore for $u \in H$ we have $y^{-2}u \in E$. Thus $y^{-2}Kv \in E$ as required.

Let $\phi \in C_0^\infty(U)$ be such that it vanishes near the x–axis. We write $\phi = \phi(z,t)$, where $z \in \Omega$ and $t \in S$, where S is the set of unit vectors in $\mathbf{R}^5$ perpendicular to the x–axis. For a given $t \in S$, $y^2\phi(.,t)$ is a test function on Ω. Let $u \in H$. In view of (7.37), we have

$$< u, y^2\phi(.,t) >_H \quad = \quad < y^{-2}u, \phi(.,t) >_E \tag{7.38}$$

$$= \int_\Omega y^2 \nabla(y^{-2}u)\nabla(\phi(z,t))d\nu(z) \tag{7.39}$$

where ∇ is the 5–dimensional gradient operator. Integrating over t, with respect to the suitably normalised Lebesgue measure σ on S yields,

$$\begin{aligned} \int_S < u, y^2\phi(.,t) >_H d\sigma(t) &= \int_S \int_\Omega y^2\nabla(y^{-2}u)\nabla(\phi(z,t))d\nu(z)d\sigma(t) &(7.40)\\ &= \int_U \nabla(y^{-2}u).\nabla\phi d\tau =< y^{-2}u, \phi >_E &(7.41)\end{aligned}$$

Now Kv is a critical point of Ψ_H^v, thus for every $h \in H$ we have

$$< Kv, h >_H = \int_\Omega vhd\nu. \tag{7.42}$$

Substituting $Kv = u$ into equations (7.40) and (7.41) and using (7.42) we obtain

$$\begin{aligned} < y^{-2}Kv, \phi >_E &= \int_S < Kv, y^2\phi(.,t) >_H d\sigma(t) &(7.43)\\ &= \int_S \int_\Omega vy^2\phi(z,t)d\nu(z)d\sigma(t) = \int_U v\phi d\tau &(7.44)\end{aligned}$$

Thus we have shown (7.34) for every test function ϕ on U that vanishes near the x–axis. We claim such functions are dense in E. Let $\varphi \in C_0^\infty(U)$. There exists increasing $\Psi \in C_0^\infty(\mathbf{R})$ satisfying

$$\Psi(s) = 0 \text{ for } s \le 1, \Psi(s) = 1 \text{ for } s \ge 2. \tag{7.45}$$

The sequence $\{\Psi(ny)\varphi\}_{n=1}^\infty$ of test functions on U, which vanish near the x–axis, converge to φ in E. This shows that (7.34) holds for all test functions φ, and it follows that (7.34) holds for $h \in E$. This completes the proof.

7.4 Maximum Principle Variant

The following theorem is based on [**20**, Theorem 8.1, pages 179–180].

Theorem 2 *(A generalised weak maximum principle for the Laplacian on an unbounded domain)*

Let $u \in W^{1,2}(\Omega)$ satisfy $\Delta u \geq 0$ (≤ 0) in the generalised sense in Ω, an unbounded domain in $\mathbf{R}^n$, where a Poincaré inequality holds. Then

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ \ (\inf_{\Omega} u \geq \inf_{\partial\Omega} u^-) \tag{7.46}$$

(where $u^+(s) = \max\{u(s), 0\}$, $u^-(s) = \min\{u(s), 0\}$.)

We elucidate some of the statements in the above theorem. We say $u \in W^{1,2}(\Omega)$ satisfies $\Delta u = 0$ ($\geq 0, \leq 0$) respectively in the generalised (or weak) sense in Ω if

$$H(u, \varphi) = \int_{\Omega} \nabla u . \nabla \varphi = 0 \ (\leq 0, \geq 0) \tag{7.47}$$

for all non-negative $\varphi \in C_0^1(\Omega)$ (the functions φ are often referred to as *test functions*).

For $u \in W^{1,2}(\Omega)$, we say u satisfies $u \leq 0$ on $\partial\Omega$ in the *weak* sense if its positive part $u^+ \in W_0^{1,2}(\Omega)$. Similarly $u \geq 0$ on $\partial\Omega$ in the *weak* sense if and only if $u^- \in W_0^{1,2}(\Omega)$. We say $u = 0$ in the *weak* sense on $\partial\Omega$ if $u \leq 0$ and $u \geq 0$ in the weak sense. For $u, v \in W^{1,2}(\Omega)$, we say $u \leq v$ in the *weak* sense on $\partial\Omega$, if $u - v \leq 0$ in the weak sense. Similarly we define

$$\sup_{\partial\Omega} u = \inf\{k | u \leq k \text{ on } \partial\Omega \text{ in the weak sense, } k \in \mathbf{R}\} \tag{7.48}$$

and

$$\inf_{\partial\Omega} u = -\sup_{\partial\Omega}(-u). \tag{7.49}$$

Proof (of the above Theorem)

We use the methods of [**20**, Theorem 8.1, pages 179–180]. Let $u \in W^{1,2}(\Omega)$ satisfy $\triangle u \geq 0$ in the generalised sense. For non–negative $v \in W_0^{1,2}(\Omega)$ we have

$$H(u,v) = \int_\Omega \nabla u.\nabla v \leq 0 \tag{7.50}$$

because v is the limit of a sequence of non–negative test functions on Ω. Put $v = (u-l)_+$, where $l = \sup_{\partial\Omega} u^+$. Then v is non–negative and a member of $W_0^{1,2}(\Omega)$ as required. Then

$$\int_{\{z|u(z)>l\}} |\nabla u|^2 = \int_{\{z|u(z)>l\}} \nabla u.\nabla v = \int_\Omega \nabla u.\nabla v \leq 0. \tag{7.51}$$

Therefore $\nabla u = 0$ almost everywhere on $\{z|u(z) > l\}$, whence u is constant on $\{z|u(z) > l\}$. We require that the weak derivative of u exists (and belongs to $L^2(\Omega)$), therefore it follows that we have either $\{z|u(z) > l\} = \Omega$ except possibly for a set of measure zero, or $\{z|u(z) > l\}$ has zero measure. However $l = \sup_{\partial\Omega} u^+$ (in the sense described above), therefore the former case cannot occur. Thus $\{z|u(z) > l\}$ has zero measure. Thus $u(z) \leq l$, for almost every $z \in \Omega$. This completes the proof.

7.5 Properties of the maximising functions

Lemma 2 *K is a positive map (with respect to the normal order on function spaces) and for non–negative, non–zero $v \in L^1(\Omega) \bigcap L^p(\Omega)$ for $p > \frac{5}{2}$, $Kv - \frac{1}{2}\lambda y^2 \in L^\infty(\Omega)$.*

Proof For v as above, K satisfies the following equations

$$\mathcal{L}Kv = v \text{ on } \Omega \tag{7.52}$$

$$Kv = 0 \text{ on } \partial\Omega \tag{7.53}$$

The proof of Lemma 1 yields that $y^{-2}Kv \in E$, which is a re–norming of $W_0^{1,2}(U)$. Identifying appropriately, we have

$$-\triangle_5(y^{-2}Kv) = v \text{ on } U \tag{7.54}$$

$$y^{-2}Kv = 0 \text{ on } \partial U \tag{7.55}$$

Applying the weak maximum principle Theorem 2 of Section 7.4 to the above problem we obtain $y^{-2}Kv \geq 0$, whence $Kv \geq 0$. Thus K is a positive operator.

$\frac{1}{2}\lambda y^2$ is clearly bounded on Ω, thus it suffices to show that $Kv \in L^\infty(\Omega)$. Lemma 1 yields that $y^{-2}Kv = \mathcal{K}v$, therefore it suffices to show that $\mathcal{K}v \in L^\infty(U)$. Applying Lemma 1 to equations (7.54) and (7.55) we have

$$-\triangle_5(\mathcal{K}v) = v \text{ on } U \tag{7.56}$$

$$\mathcal{K}v = 0 \text{ on } \partial U \tag{7.57}$$

We write $\triangle$ to denote the 5–dimensional Laplacian for the remainder of this proof. For $z, w \in \mathbf{R}^5$ we define

$$\tilde{K}v(z) = \int_U G(z,w)v(w)d\tau(w) \tag{7.58}$$

where

$$G(z,w) = \frac{1}{8\pi^2|z-w|^3} \tag{7.59}$$

and $|.|$ denotes Euclidean distance in $\mathbf{R}^5$. Note that $\tilde{K}v(z) \geq 0$ for every $z \in \mathbf{R}^5$. We aim to show that $\mathcal{K}v(z) \leq \tilde{K}v(z)$ for almost every $z \in U$. [**20**] yields no regularity results for the operator $\tilde{K}$ on an unbounded domain, therefore we work on bounded domains and seek appropriate limits. For $n \in \mathbf{N}$, we define

$U_n = B_n(0) \bigcap U$, where $B_n(0)$ denotes the open ball of radius n in $\mathbf{R}^5$, centred at the origin. U_n is a bounded subdomain of U. Define E_n as the completion of the test functions on U_n with respect to the scalar product

$$< u, v >_{E_n} = \int_{U_n} \nabla u . \nabla v d\tau \tag{7.60}$$

where the measure τ has Radon–Nikodym derivative (or density) $\frac{1}{\pi}$ with respect to 5–dimensional Lebesgue measure. E_n is a re-norming of $H_0^1(U_n)$. For $n \in \mathbf{N}$, define $K_n v$ as the unique minimiser over $u \in E_n$ of the functional

$$\Psi_n(u) = \frac{1}{2} \int_{U_n} |\nabla u|^2 d\tau - \int_{U_n} uv d\tau. \tag{7.61}$$

The following equations are satisfied

$$-\triangle K_n v = v \text{ on } U_n \tag{7.62}$$

$$K_n v = 0 \text{ on } \partial U_n \tag{7.63}$$

where $K_n v = 0$ on ∂U_n in the weak sense (because $K_n v \in H_0^1(U_n)$ by the definition of the operator K_n) and regularity theory [**20**, Theorem 8.9, page 185] shows that $K_n v \in W^{1,2}(U_n) \bigcap W_{loc}^{2,2}(U_n)$.

For each $n \in \mathbf{N}$, a special case of the Calderon–Zygmund inequality [**20**, Theorem 9.9, page 230] yields that $\tilde{K} v_n \in W^{2,2}(U_n)$, where v_n denotes $v|_{U_n}$. From equation (7.58) and [**20**, Theorem 9.9, page 230] we obtain

$$-\triangle \tilde{K} v_n = v \text{ on } U_n \tag{7.64}$$

$$\tilde{K} v_n \geq 0 \text{ on } \partial U_n \tag{7.65}$$

where equation (7.64) holds almost everywhere, and $\tilde{K}v_n \geq 0$ on ∂U_n in the weak sense (because $\tilde{K}v_n(z) \geq 0$ for all $z \in U_n$ by definition). Combining (7.62)-(7.63) and (7.64)–(7.65) we obtain

$$-\Delta(\tilde{K}v_n - K_n v) = 0 \text{ on } U_n \tag{7.66}$$

$$\tilde{K}v_n - K_n v \geq 0 \text{ on } \partial U_n \tag{7.67}$$

where the latter equation holds in the weak sense. By way of explanation we note that $\tilde{K}v_n \in W^{2,2}(U_n)$, $(\tilde{K}v_n)^- \in H_0^1(U_n)$, and $K_n v \in H_0^1(U_n)$, therefore we have $(\tilde{K}v_n - K_n v)^- \in H_0^1(U_n)$ (verified in the Appendix), which is exactly the statement that $\tilde{K}v_n - K_n v \geq 0$ on ∂U_n in the weak sense. In addition we see that $\tilde{K}v_n - K_n v \in W^{1,2}(U_n)$, and applying a generalised weak maximum principle [**20**, Theorem 8.1, page 179] we obtain that $\inf_{U_n}(\tilde{K}v_n - K_n v) \geq \inf_{\partial U_n}(\tilde{K}v_n - K_n v)^- = 0$, whence

$$\tilde{K}v_n(z) \geq K_n v(z) \tag{7.68}$$

for almost every $z \in U_n$, and every $n \in \mathbb{N}$, whence

$$\tilde{K}v(z) \geq K_n v(z) \tag{7.69}$$

for almost every $z \in U_n$, and each $n \in \mathbb{N}$.

We show that $\{K_n v\}_{n=1}^{\infty}$ is a minimising sequence for Ψ over E. (Note that $K_n v \in E$ by extending the function to be zero outside U_n). Let $\epsilon > 0$. Define

$$G_\epsilon = \left\{ u \in E \,|\, \Psi(u) < \inf_{u \in E} \Psi(u) + \epsilon \right\}. \tag{7.70}$$

G_ϵ is non–empty (by the definition of infimum), and open (by the continuity of Ψ). $C_0^\infty(U)$ functions are dense in E, therefore there exists $\theta_n \in C_0^\infty \bigcap G_\epsilon$, where supp $\theta_n \subset U_n$, for some $n \in \mathbb{N}$. Now $\inf_{u \in E_n} \Psi(u)$ is a decreasing function of n, therefore we have,

$$\Psi(K_m v) \leq \Psi(\theta_n) < \inf_{u \in E} \Psi(u) + \epsilon, \text{ for all } m \geq n. \tag{7.71}$$

Thus $\{K_n v\}_{n=1}^{\infty}$ is a minimising sequence for Ψ, whence

$$\Psi(K_n v) \to \inf_{u \in E} \Psi(u) = \Psi(\mathcal{K} v) \text{ as } n \to \infty. \tag{7.72}$$

Using the methods of Lemma 1 of Chapter 6, noting that Ψ is coercive, strictly convex and weakly lower semicontinuous, we pass to a subsequence if necessary to obtain

$$K_n v \xrightarrow{w} \mathcal{K} v \text{ as } n \to \infty. \tag{7.73}$$

It may be shown that

$$\int_U v \mathcal{K} v d\tau = ||\mathcal{K} v||_E^2 \tag{7.74}$$

$$\int_{U_n} v K_n v d\tau = ||K_n v||_{E_n}^2 = ||K_n v||_E^2 \tag{7.75}$$

for v as above. Therefore (7.72)–(7.75) gives that

$$||K_n v||_E \to ||\mathcal{K} v||_E \text{ as } n \to \infty. \tag{7.76}$$

Combining (7.73) and (7.76) we obtain $K_n v \to \mathcal{K} v$ strongly in E, whence $K_n v \to \mathcal{K} v$ strongly in $H_0^1(U)$. By the corollary to the Riesz–Fischer Theorem, $K_n v \to \mathcal{K} v$ pointwise almost everywhere. Recalling that

$$K_n v(z) \leq \tilde{K} v(z) \text{ for almost every } z \in U_n, \text{ all } n \in \mathbb{N} \tag{7.77}$$

we have

$$\mathcal{K} v(z) \leq \tilde{K} v(z) \text{ for almost every } z \in U \tag{7.78}$$

as required.

We change to polar co-ordinates about z. Let $\rho = |z - w|$. Then by (7.78) we have

$$\begin{aligned}
\mathcal{K}v(z) &\leq \int_U \frac{1}{8\pi^2|z-w|^3} v(w)dw & (7.79)\\
&= \int_{\{w\in U|\rho<1\}} \frac{1}{8\pi^2|z-w|^3} vdw + \int_{\{w\in U|\rho\geq 1\}} \frac{1}{8\pi^2|z-w|^3} vdw & (7.80)
\end{aligned}$$

Firstly,

$$\int_{\{w\in U|\rho\geq 1\}} G(z,w)v(w)dw \leq \frac{1}{8\pi^2}||v||_1. \tag{7.81}$$

Further

$$\begin{aligned}
&\int_{\{w\in U|\rho<1\}} \frac{1}{8\pi^2|z-w|^3} v(w)dw & (7.82)\\
\leq\ & ||v||_p \left\{ \int_{\{w\in\Omega|\rho<1\}} \left| \frac{1}{8\pi^2|z-w|^3} \right|^q dw \right\}^{\frac{1}{q}} & (7.83)\\
=\ & \kappa_1 ||v||_p \left\{ \int_0^1 \left(\frac{1}{|\rho^3|} \right)^q \rho^4 d\rho \right\}^{\frac{1}{q}} & (7.84)\\
=\ & \kappa_1 ||v||_p \left\{ \int_0^1 \rho^{4-3q} d\rho \right\}^{\frac{1}{q}} = \kappa_2 ||v||_p. & (7.85)
\end{aligned}$$

for some constants κ_1, κ_2, noting that $4 - 3q > -1$ by our choice of p (q denotes the conjugate exponent of p). Combining equations (7.79)–(7.85) we have

$$\mathcal{K}v(z) \leq \frac{1}{8\pi^2}||v||_1 + \kappa_2||v||_p. \tag{7.86}$$

Thus $\mathcal{K}v \in L^\infty(U)$, whence $Kv \in L^\infty(\Omega)$. This completes the proof.

Lemma 3 *Let non-negative $f_0 \in L^1(\Omega) \bigcap L^p(\Omega)$, for $p > \frac{5}{2}$. Suppose, for a given $\lambda > 0$, Ψ_λ has a maximiser $\tilde{f}$ relative to $R(f_0)$. Then $\tilde{u} = K\tilde{f}$ satisfies*

$\mathcal{L}\tilde{u} = \phi(\tilde{u} - \frac{1}{2}\lambda y^2)$ *in the weak sense, for some increasing function* ϕ.

Proof Recall, for $\lambda > 0$ and non–negative $v \in L^1(\Omega) \bigcap L^p(\Omega)$, for p as above,

$$\Psi_\lambda(v) = \frac{1}{2}\int_\Omega vKv d\nu - \frac{1}{2}\lambda\int_\Omega vy^2 d\nu. \tag{7.87}$$

Let $h \in L^1(\Omega) \bigcap L^p(\Omega)$ (we work in the space $L^1(\Omega) \bigcap L^p(\Omega)$ with the norm $||.|| = ||.||_1 + ||.||_p$), then

$$\begin{aligned} & \Psi_\lambda(v+h) - \Psi_\lambda(v) & (7.88) \\ = \; & \frac{1}{2}\int_\Omega (v+h)K(v+h)d\nu - \frac{1}{2}\lambda\int_\Omega (v+h)y^2 d\nu & \\ & -\frac{1}{2}\int_\Omega vKvd\nu + \frac{1}{2}\lambda\int_\Omega vy^2 d\nu & (7.89) \\ = \; & \int_\Omega hKvd\nu + \frac{1}{2}\int_\Omega hKhd\nu - \frac{1}{2}\lambda\int_\Omega hy^2 d\nu & (7.90) \end{aligned}$$

where we have used the fact that

$$\int_\Omega uKvd\nu = \int_\Omega vKud\nu \tag{7.91}$$

for non–negative $v, u \in L^1(\Omega) \bigcap L^p(\Omega)$. (This was proved in Lemma 2 of Chapter 6). Then the Fréchet derivative of Ψ_λ, which we will denote $d\Psi_\lambda[v]$, is given by

$$d\Psi_\lambda[v] = Kv - \frac{1}{2}\lambda y^2. \tag{7.92}$$

By way of explanation, we note that $d\Psi_\lambda[v] \in L^\infty(\Omega)$, by Lemma 2. Therefore $d\Psi_\lambda[v]$ belongs to the dual of the space $L^1(\Omega) \bigcap L^p(\Omega)$. Ψ_λ is a strictly convex and Fréchet differentiable function, thus Ψ_λ is sub–differentiable at v, for v as above, and $\partial\Psi_\lambda(v) = \{d\Psi_\lambda[v]\}$. Therefore, for every $f \in R(f_0)\backslash\{\tilde{f}\}$ we have

$$\Psi_\lambda(\tilde{f}) \geq \Psi_\lambda(f) > \int_\Omega (K\tilde{f} - \frac{1}{2}\lambda y^2)(f - \tilde{f})d\nu + \Psi_\lambda(\tilde{f}) \tag{7.93}$$

where the strict inequality follows by the strict convexity of Ψ_λ . Rearranging, we obtain

$$\int_\Omega (K\tilde{f} - \frac{1}{2}\lambda y^2) f d\nu < \int_\Omega (K\tilde{f} - \frac{1}{2}\lambda y^2)\tilde{f} d\nu, \ \forall f \in R(f_0)\backslash\{\tilde{f}\}. \tag{7.94}$$

We now apply Theorem 1 to $\tilde{f} \in L^1(\Omega)$ and $K\tilde{f} - \frac{1}{2}\lambda y^2 \in L^\infty(\Omega)$ to obtain

$$\tilde{f} = \phi(K\tilde{f} - \frac{1}{2}\lambda y^2) \tag{7.95}$$

for some increasing function ϕ, ν–almost everywhere. Now $K\tilde{f} = \tilde{u}$ for some $\tilde{u} \in H^1_0(\Omega)$. Accordingly, we obtain

$$\mathcal{L}\tilde{u} = \phi(\tilde{u} - \frac{1}{2}\lambda y^2) \tag{7.96}$$

in the weak sense. This completes the proof.

Lemma 4 *Let non–negative $f_0 \in L^p(\Omega) \bigcap L^1(\Omega)$, for $p > \frac{5}{2}$. Let $\lambda > 0$. Let $\tilde{f}$ be a maximiser for Ψ_λ relative to $\overline{R(f_0)}^w$. (We know such a maximiser exists). Then*

$$\tilde{f}^{-1}(0,\infty) \subset (K\tilde{f} - \frac{1}{2}\lambda y^2)^{-1}(0,\infty) \tag{7.97}$$

except possibly for a set of measure zero. Moreover

$$\nu(\tilde{f}^{-1}(0,\infty)) \leq \nu\left((K\tilde{f} - \frac{1}{2}\lambda y^2)^{-1}(0,\infty)\right) \tag{7.98}$$

(when the above expression makes sense)

Proof The proof of Lemma 3 yields that Ψ_λ is sub–differentiable at $\tilde{f}$, and that $\partial\Psi_\lambda(\tilde{f}) = \{K\tilde{f} - \frac{1}{2}\lambda y^2\}$. Therefore, for every $f \in \overline{R(f_0)}^w \backslash \{\tilde{f}\}$ we have

$$\Psi_\lambda(\tilde{f}) \geq \Psi_\lambda(f) > \int_\Omega (K\tilde{f} - \frac{1}{2}\lambda y^2)(f - \tilde{f}) d\nu + \Psi_\lambda(\tilde{f}) \tag{7.99}$$

where the strict inequality follows by the strict convexity of Ψ_λ. Rearranging, we obtain

$$\int_\Omega (K\tilde{f} - \frac{1}{2}\lambda y^2) f d\nu < \int_\Omega (K\tilde{f} - \frac{1}{2}\lambda y^2)\tilde{f} d\nu, \ \forall f \in \overline{R(f_0)}^w \backslash \{\tilde{f}\}. \tag{7.100}$$

Let $S = \tilde{f}^{-1}(0,\infty)$. We suppose (7.97) is false, for a contradiction. Then there exists $A \subset S$, a set of positive measure such that $K\tilde{f} - \frac{1}{2}\lambda y^2 \leq 0$ on A. Define

$$\bar{f}(z) = \begin{cases} 0 & z \in A \\ \tilde{f}(z) & z \in \Omega \backslash A \end{cases}$$

for $z \in \Omega$. Then $\bar{f} \in \overline{R(f_0)}^w$ (by Theorem 2 of Chapter 3), $\bar{f} \neq \tilde{f}$, and

$$\int_\Omega (K\tilde{f} - \frac{1}{2}\lambda y^2)\bar{f} d\nu \geq \int_\Omega (K\tilde{f} - \frac{1}{2}\lambda y^2)\tilde{f} d\nu \tag{7.101}$$

which contradicts (7.100). This establishes (7.97), and the result follows. This completes the proof.

Lemma 5 *Let non-negative $f_0 \in L^1(\Omega) \bigcap L^p(\Omega)$, where $p > \frac{5}{2}$. Fix $\lambda > 0$. Let $\tilde{f}$ be a maximiser for Ψ_λ relative to $\overline{R(f_0)}^w$. Suppose that $\tilde{f} \notin R(f_0)$. Then*

$$(K\tilde{f} - \frac{1}{2}\lambda y^2)^{-1}(0,\infty) = \tilde{f}^{-1}(0,\infty) \tag{7.102}$$

except possibly for a set of measure zero.

Proof In Lemma 4 it was shown that

$$\tilde{f}^{-1}(0,\infty) \subset (K\tilde{f} - \frac{1}{2}\lambda y^2)^{-1}(0,\infty) \tag{7.103}$$

except possibly for a set of measure zero. We show that

$$(K\tilde{f} - \frac{1}{2}\lambda y^2)^{-1}(0,\infty) \subset \tilde{f}^{-1}(0,\infty) \tag{7.104}$$

except possibly for a set of measure zero. We suppose the statement is false, to show a contradiction. Therefore there exists $V^1 \subset (K\tilde{f} - \frac{1}{2}\lambda y^2)^{-1}(0,\infty)$, a set of positive measure, such that $V^1 \subset \tilde{f}^{-1}(0)$. We have previously seen that $\tilde{f} \in RC(f_0)$, and by assumption $\tilde{f} \notin R(f_0)$. The definition of $RC(f_0)\backslash R(f_0)$ gives the existence of $\alpha > \beta > 0$ such that $\nu\left(f_0^{-1}(\beta,\alpha]\right) > \nu\left(\tilde{f}^{-1}(\beta,\alpha]\right)$. Let $\delta = \nu\left(f_0^{-1}(\beta,\alpha]\right) - \nu\left(\tilde{f}^{-1}(\beta,\alpha]\right)$. Let $V \subset V^1$, where $0 < \nu(V) \leq \delta$. (The existence of such a V is established by arguments similar to those used in the proof of Theorem 1.) Define

$$\hat{f}(x) = \begin{cases} \tilde{f}(x) & \text{if } x \in \Omega\backslash V \\ \beta & \text{if } x \in V \end{cases}$$

Now $\hat{f}$ is non–negative, and $\hat{f} \neq \tilde{f}$. Moreover, for any positive real number σ,

$$\int_\Omega (\hat{f} - \sigma)_+ d\nu \leq \int_\Omega (f_0 - \sigma)_+ d\nu. \tag{7.105}$$

The characterisation of the weak closure of the set of rearrangements given in Theorem 2 of Chapter 3, yields that $\hat{f} \in \overline{R(f_0)}^w$. Recall, from the proof of Lemma 4, that

$$\int_\Omega (K\tilde{f} - \frac{1}{2}\lambda y^2) f d\nu < \int_\Omega (K\tilde{f} - \frac{1}{2}\lambda y^2)\tilde{f} d\nu \tag{7.106}$$

for every $f \in \overline{R(f_0)}^w \backslash \{\tilde{f}\}$. However $\hat{f} \in \overline{R(f_0)}^w$, and $\hat{f} \neq \tilde{f}$, and further

$$\begin{aligned} \int_\Omega (K\tilde{f} - \frac{1}{2}\lambda y^2)\tilde{f} d\nu &= \int_{\Omega\backslash V} (K\tilde{f} - \frac{1}{2}\lambda y^2)\tilde{f} d\nu & (7.107) \\ &= \int_{\Omega\backslash V} (K\tilde{f} - \frac{1}{2}\lambda y^2)\hat{f} d\nu & (7.108) \\ &< \int_\Omega (K\tilde{f} - \frac{1}{2}\lambda y^2)\hat{f} d\nu. & (7.109) \end{aligned}$$

This contradicts (7.106), thus establishing (7.104). This completes the proof.

Corollary 1 *Let non-negative $f_0 \in L^1(\Omega) \bigcap L^p(\Omega)$, where $p > \frac{5}{2}$. Fix $\lambda > 0$. Let $\tilde{f}$ be a maximiser for Ψ_λ relative to $\overline{R(f_0)}^w$. Suppose that $\tilde{f} \notin R(f_0)$. Then $(K\tilde{f} - \frac{1}{2}\lambda y^2)^{-1}(0)$ has zero measure.*

Proof For $f \in \overline{R(f_0)}^w \backslash \{\tilde{f}\}$ we have

$$\int_\Omega (K\tilde{f} - \frac{1}{2}\lambda y^2) f d\nu < \int_\Omega (K\tilde{f} - \frac{1}{2}\lambda y^2)\tilde{f} d\nu. \tag{7.110}$$

If $(K\tilde{f} - \frac{1}{2}\lambda y^2)^{-1}(0)$ has positive measure, then the methods of the previous lemma yield the existence of $\hat{f} \in \overline{R(f_0)}^w$, $\hat{f} \neq \tilde{f}$, such that

$$\int_\Omega (K\tilde{f} - \frac{1}{2}\lambda y^2)\tilde{f} d\nu = \int_\Omega (K\tilde{f} - \frac{1}{2}\lambda y^2)\hat{f} d\nu \tag{7.111}$$

contradicting (7.110). This completes the proof.

Lemma 6 *Let non-negative non-zero $f_0 \in L^1(\Omega) \bigcap L^p(\Omega)$ for $p > \frac{5}{2}$. Let $\tilde{f}$ be a maximiser for Ψ_λ relative to $\overline{R(f_0)}^w$. Then for a given $\epsilon > 0$, there exists $M > 0$ such that for $|x| > M$, we have*

$$0 \leq K\tilde{f}(x, y) < \epsilon y^2. \tag{7.112}$$

Proof Recall the definition of the operator $\mathcal{K}$ from Section 7.3. For non-negative $v \in L^1(\Omega) \bigcap L^p(\Omega)$ for $p > \frac{5}{2}$, Lemma 1 yields that $\mathcal{K}v = y^{-2}Kv$. Therefore it suffices to show that

$$\mathcal{K}v(x, y) \to 0 \tag{7.113}$$

as $x \to \infty$, making the appropriate identification between the strip Ω and the cylinder U. Let $z, w \in U$. Identify $z, w \in U$ with $(x_1, y_1), (x_2, y_2) \in \Omega$, respectively. Let $p \geq p_1 > \frac{5}{2}$, and $1 < p_2 < \frac{5}{2}$, and let q_1, q_2 denote the conjugate exponents of p_1, p_2 respectively. For $|x_1| > 2M$, (where $M > 1$) and writing $\rho = |z - w|$, using the proof of Lemma 2 we have

$$
\begin{aligned}
\mathcal{K}v(x_1, y_1) &\leq \int_U \frac{1}{8\pi^2|z-w|^3} v(w)dw && (7.114)\\
&= \int_{\{w\in U|\rho<1\}} \frac{1}{8\pi^2\rho^3} v(w)dw \\
&+ \int_{\{w\in U|1\leq\rho\leq M\}} \frac{1}{8\pi\rho^3} v(w)dw + \int_{\{w\in U|\rho>M\}} \frac{1}{8\pi^2\rho^3} v dw && (7.115)\\
&\leq \kappa_1 ||v|_{x_2>M}||_{p_1} \left\{\int_0^1 \rho^{4-3q_1} d\rho\right\}^{\frac{1}{q_1}} \\
&\quad +\kappa_2 ||v|_{x_2>M}||_{p_2} \left\{\int_1^M \rho^{4-3q_2} d\rho\right\}^{\frac{1}{q_2}} + \frac{1}{8\pi M^3}||v||_1 && (7.116)\\
&\leq \kappa_3 ||v|_{x_2>M}||_{p_1} + \kappa_4 ||v|_{x_2>M}||_{p_2} + \frac{1}{8\pi^2 M^3}||v||_1. && (7.117)
\end{aligned}
$$

$\kappa_1, \kappa_2, \kappa_3$ and κ_4 are constants, noting that $\int_0^1 \rho^{4-3q_1} d\rho$ and $\int_1^\infty \rho^{4-3q_2} d\rho$ are finite by our choice of p_1 and p_2. Now we have

$$|\mathcal{K}v(x_1, y_1)| \leq \kappa_3 ||v|_{x_2>M}||_{p_1} + \kappa_4 ||v|_{x_2>M}||_{p_2} + \frac{1}{8\pi^2 M^3}||v||_1 \qquad (7.118)$$

for $x_1 > 2M$ (where $M > 1$) and the right hand side tends to zero as $M \to \infty$. Now apply (7.118) for $v = \tilde{f}$. This completes the proof.

Theorem 3 *Let non-negative $f_0 \in L^1(\Omega) \bigcap L^p(\Omega)$, for $p > \frac{5}{2}$. Define, for $\lambda > 0$,*

$$\Psi_\lambda(w) = \frac{1}{2}\int_\Omega wKw d\nu - \frac{1}{2}\lambda \int_\Omega wy^2 d\nu. \qquad (7.119)$$

For fixed $\lambda > 0$, let $\tilde{f}$ be a maximiser for Ψ_λ relative to $\overline{R(f_0)}^w$ (we know such an $\tilde{f}$ exists). Then the following are true

(i) $\tilde{u} = K\tilde{f}$ satisfies

$$\mathcal{L}\tilde{u} = \phi(\tilde{u} - \frac{1}{2}\lambda y^2) \qquad (7.120)$$

in the weak sense, for some increasing function ϕ.

(ii) Except possibly for a set of measure zero,

$$\tilde{f}^{-1}(0,\infty) \subset (K\tilde{f} - \frac{1}{2}\lambda y^2)^{-1}(0,\infty) \tag{7.121}$$

and $(K\tilde{f} - \frac{1}{2}\lambda y^2)^{-1}[0,\infty)$ *is a bounded set, whence* $\tilde{f}^{-1}(0,\infty)$ *is also bounded. Moreover*

$$\nu(\tilde{f}^{-1}(0,\infty)) \leq \nu((K\tilde{f} - \frac{1}{2}\lambda y^2)^{-1}(0,\infty)) < \infty \tag{7.122}$$

and in particular, if the set $f_0^{-1}(0,\infty)$ *has infinite* ν*–measure, then there exists no maximiser for* Ψ_λ *relative to* $R(f_0)$.

(iii) Suppose $\tilde{f} \notin R(f_0)$. *Then*

$$(K\tilde{f} - \frac{1}{2}\lambda y^2)^{-1}(0,\infty) = \tilde{f}^{-1}(0,\infty) \tag{7.123}$$

except possibly for a set of ν*–measure zero, and* $(K\tilde{f} - \frac{1}{2}\lambda y^2)^{-1}(0)$ *has zero measure.*

Proof For $\tilde{f} \in R(f_0)$, (i) follows by Lemma 3. Now $\tilde{f} \in R(\tilde{f})$, and is a maximiser for Ψ_λ relative to $R(\tilde{f})$ (noting $R(\tilde{f}) \subset \overline{R(f_0)}^w$). Therefore we can apply Lemma 3 to obtain (i).

Lemma 4 yields that

$$\tilde{f}^{-1}(0,\infty) \subset (K\tilde{f} - \frac{1}{2}\lambda y^2)^{-1}(0,\infty) \tag{7.124}$$

except possibly for a set of measure zero. Lemma 6 shows the existence of $M > 0$ such that for $|x| > M$,

$$0 \leq K\tilde{f}(x,y) < \frac{1}{2}\lambda y^2 \tag{7.125}$$

whence $K\tilde{f}(x,y)-\frac{1}{2}\lambda y^2<0$ for $|x|>M$. Therefore the set $(K\tilde{f}-\frac{1}{2}\lambda y^2)^{-1}(0,\infty)$ is bounded. Thus $\tilde{f}^{-1}(0,\infty)$ is bounded. Taking the measure of the sets in (7.124) yields (7.122), and further if $f_0^{-1}(0,\infty)$ has infinite measure, then there exists no maximiser for Ψ_λ relative to the set of rearrangements. This shows (ii).

(iii) follows by Lemma 5 and Corollary 1. This completes the proof.

Theorem 4 *Let non-negative $f_0 \in L^1(\Omega)\bigcap L^p(\Omega)$, for $p>\frac{5}{2}$. Define, for $\lambda>0$,*

$$\Psi_\lambda(v)=\frac{1}{2}\int_\Omega vKv\,d\nu-\frac{1}{2}\lambda\int_\Omega vy^2d\nu \tag{7.126}$$

Fix $\lambda>0$. Let $\tilde{f}$ be a maximiser for Ψ_λ relative to $\overline{R(f_0)}^w$. Then there exists a sequence $\{\xi_n\}_{n=1}^\infty$ with $\xi_n \in R(f_0)$ for each $n\in\mathbb{N}$, with weak limit $\tilde{f}$ in $L^p(\Omega)$, and such that $\Psi_\lambda(\xi_n)\to\Psi_\lambda(\tilde{f})$.

Proof Theorem 3 (ii) yields that $\tilde{f}^{-1}(0,\infty)$ is bounded. We choose $M>0$ such that $\tilde{f}^{-1}(0,\infty)\subset[-M,M]\times(0,R)=\tilde{M}$, say. Further, from Theorem 1 of Chapter 6 we have $\tilde{f}\in RC(f_0)$. We can find $\xi_0\in R(f_0)$ such that $\xi_0|_{\tilde{M}}=\tilde{f}|_{\tilde{M}}$. We write $M_0=\Omega\backslash\tilde{M}$. Define, for $n\in\mathbb{N}$,

$$M_n=(-\infty,-M-n)\bigcup(M+n,\infty)\times\left(0,\frac{R}{n}\right) \tag{7.127}$$

For each $n\in\mathbb{N}$, Theorem 1 of Chapter 3 yields the existence of a measure preserving transformation $T_n: M_n\to M_0$. We define, for each $n\in\mathbb{N}$,

$$\xi_n(z)=\begin{cases}\tilde{f}(z) & \text{if } z\in\tilde{M}\\ \xi_0\circ T_n(z) & \text{if } z\in M_n\\ 0 & \text{if } z\notin\tilde{M}\bigcup M_n\end{cases}$$

for $z\in\Omega$. The measure preserving properties of T_n ensure $\xi_n\in R(f_0)$ for each $n\in\mathbb{N}$. Let $g\in L^q(\Omega)$, where q denotes the conjugate exponent of p. Then

$$\left|\int_\Omega(\xi_n-\tilde{f})g\,d\nu\right|\le||\xi_0||_p||g|_{|z|>M+n}||_q\to 0 \tag{7.128}$$

as $n \to \infty$. Thus $\xi_n \xrightarrow{w} \tilde{f}$ in $L^p(\Omega)$.

Define, for non-negative $v \in L^1(\Omega) \bigcap L^p(\Omega)$, p as above,

$$T(v) = \frac{1}{2} \int_{\Omega} vKv d\nu. \tag{7.129}$$

T is (strongly) continuous (using Lemma 1 of Chapter 6, and Holder's inequality) and T is convex (from the proof of Corollary 1, Chapter 6). Therefore T is weakly sequentially lower semicontinuous. Further,

$$\left| \int_{\Omega} \xi_n y^2 d\nu - \int_{\Omega} \tilde{f} y^2 d\nu \right| = \left| \int_{\Omega} (\xi_n - \tilde{f}) y^2 d\nu \right| \le \frac{R^2}{n^2} ||\xi_0||_1 \to 0 \tag{7.130}$$

as $n \to \infty$. Therefore we have

$$\begin{aligned}
\Psi_\lambda(\tilde{f}) &= \frac{1}{2} \int_{\Omega} \tilde{f} K \tilde{f} d\nu - \frac{1}{2} \lambda \int_{\Omega} \tilde{f} y^2 d\nu && (7.131)\\
&\le \liminf_{n\to\infty} \frac{1}{2} \int_{\Omega} \xi_n K \xi_n d\nu - \lim_{n\to\infty} \frac{1}{2} \lambda \int_{\Omega} \xi_n y^2 d\nu && (7.132)\\
&= \liminf_{n\to\infty} \Psi_\lambda(\xi_n). && (7.133)
\end{aligned}$$

However $\tilde{f}$ is a maximiser of Ψ_λ relative to $\overline{R(f_0)}^w$, therefore $\Psi_\lambda(\xi_n) \to \Psi_\lambda(\tilde{f})$ as required. This completes the proof.

Chapter 8

Vortices in a Channel and sets of Rearrangements

8.1 Introduction

We consider a variational problem, arising from the work of Benjamin [**3**], motivated by the boundary value problem for a steady vortex in an ideal fluid flowing along an infinite channel. The variational functional is shown to attain a maximum relative to the weak closure of the set of rearrangements of a fixed function, for all positive values of a parameter. The results obtained in Chapter 3 enable this maximiser to be characterised. We establish some properties of the maximising functions.

We work in a plane infinite strip, therefore we define

$$\Omega = \{(x,y) \in \mathbb{R}^2 | 0 < y < R\} \tag{8.1}$$

where $R > 0$ is fixed, and we endow Ω with plane Lebesgue measure.

A differential operator $\mathcal{L}$ is defined on Ω by

$$\mathcal{L}u = -\Delta u \tag{8.2}$$

where Δu represents the Laplacian of u in 2 dimensions.

For a solution u to the problem to be studied, where u represents the stream function, the *vortex core* is the region where the vorticity $\mathcal{L}u > 0$. At infinity the fluid velocity approaches a uniform stream of speed λ relative to the vortex core.

We define an inverse K for $\mathcal{L}$ satisfying

$$\mathcal{L}Kv = v \text{ on } \Omega \tag{8.3}$$

$$Kv = 0 \text{ on } \partial\Omega \tag{8.4}$$

for $v \in L^p(\Omega)$, for suitable p.(See Lemma 1). $\partial\Omega$ represents the boundary of Ω.

When $\lambda > 0$, for $v \in L^p(\Omega)$, a variational functional is defined by

$$\varphi_\lambda(v) = \frac{1}{2}\int_\Omega vKv d\mu - \lambda \int_\Omega vy d\mu \tag{8.5}$$

where μ denotes plane Lebesgue measure. The former term of the functional represents energy and the latter momentum.

We prove the following theorem;

Theorem 1 Let f_0 be non–zero and non–negative, $f_0 \in L^1(\Omega) \bigcap L^p(\Omega)$, $1 < p < 2$. Then for any $\lambda > 0$,

(i) φ_λ attains a maximum value relative to $\overline{R(f_0)}^w$.

(ii) All maximisers of φ_λ relative to $\overline{R(f_0)}^w$ are members of $RC(f_0)$.

The method of proof is to construct a maximising sequence for φ_λ relative to $\overline{R(f_0)}^w$, with each function Steiner symmetric (the definition is given in Section 6.3). $\overline{R(f_0)}^w$ is weakly compact, therefore the sequence has a subsequential weak limit (which is Steiner symmetric). The weak upper semicontinuity of φ_λ with respect to the Steiner symmetric elements of $\overline{R(f_0)}^w$ yields the existence of a maximiser. We have not been able to show that the maximiser is unique.

Theorem 2 Let non-negative $f_0 \in L^1(\Omega) \bigcap L^p(\Omega)$ for $p > 2$. Fix $\lambda > 0$. Let $\tilde{f}$ be a maximiser for φ_λ relative to $\overline{R(f_0)}^w$. Then

(i) If $\tilde{f} \in R(f_0)$, then there exists $\tilde{u}(= K\tilde{f})$ such that

$$- \Delta \tilde{u} = \phi(\tilde{u} - \lambda y) \tag{8.6}$$

in the weak sense, for some increasing function ϕ.

(ii) Except possibly for a set of measure zero,

$$\tilde{f}^{-1}(0, \infty) \subset (K\tilde{f} - \lambda y)^{-1}(0, \infty). \tag{8.7}$$

Further $(K\tilde{f}-\lambda y)^{-1}(0,\infty)$ is a bounded set, whence the set $\tilde{f}^{-1}(0,\infty)$ is bounded.

(iii) If $\tilde{f} \notin R(f_0)$,

$$(K\tilde{f} - \lambda y)^{-1}(0, \infty) = \tilde{f}^{-1}(0, \infty) \tag{8.8}$$

except possibly for a set of measure zero, and $(K\tilde{f} - \lambda y)^{-1}(0)$ has zero measure.

We show that the maximiser $\tilde{f}$ is the strict maximiser (relative to $R(f_0)$ or $\overline{R(f_0)}^w$ as appropriate) of a certain linear functional. We apply Theorem 1 of Chapter 7 to obtain (i), and then use the fact that $\tilde{f}$ is a strict maximiser to obtain (ii) and (iii). An upper bound for the size of the set $(K\tilde{f} - \lambda y)^{-1}(0, \infty)$ is established by using Newtonian potentials.

We apply the above theorems to obtain the following;

Theorem 3 Let non-negative $f_0 \in L^1(\Omega) \bigcap L^p(\Omega)$ for $p > 2$. Fix $\lambda > 0$. Let $\tilde{f}$ be a maximiser for φ_λ relative to $\overline{R(f_0)}^w$. Then there exists a sequence $\{\xi_n\}_{n=1}^\infty$ in $R(f_0)$ for each $n \in \mathbb{N}$, $\{\xi_n\}_{n=1}^\infty$ has weak limit $\tilde{f}$, and $\varphi_\lambda(\xi_n) \to \varphi_\lambda(\tilde{f})$.

Thus, for $\lambda > 0$, for a maximiser $\tilde{f}$ for φ_λ relative to $\overline{R(f_0)}^w$, we can find a maximising sequence of rearrangements for φ_λ with weak limit $\tilde{f}$. The theorem shows that $\sup_{f\in R(f_0)}\varphi_\lambda(f) = \sup_{f\in \overline{R(f_0)}^w}\varphi_\lambda(f)$. However we can find weakly convergent maximising sequences relative to $R(f_0)$ or $\overline{R(f_0)}^w$, where the weak limit is not a maximiser.

The proof of the theorems stated above requires some results concerning Steiner symmetrisation. These are recalled in the next section.

8.2 Steiner Symmetrisation

Let non-negative $v \in L^p(\Omega)$, $1 < p < \infty$. We recall the definition of the *Steiner symmetrisation* v^* of v,(with respect to the line $x = 0$), given in Section 6.3. For non-negative $f, g \in L^p(\Omega)$, p as above, and non-negative $h \in L^q(\Omega)$ (where q denotes the conjugate exponent of p) we have

$$||f^* - g^*||_p \leq ||f - g||_p \tag{8.9}$$

$$\int_\Omega fh\,d\mu \leq \int_\Omega f^*h^*\,d\mu \tag{8.10}$$

We state a result on Steiner symmetrisation from Appendix I of Fraenkel and Berger [18], where v^* was defined as above for non-negative continuous functions v with compact supports, and then defined for a general non-negative L^2 function by approximation in the 2-norm. It is easily verified that the two definitions are equivalent for non-negative $v \in L^2(\Omega)$. Fraenkel and Berger studied functions defined on a half-plane, but their results are equally applicable to functions on the strip Ω.

For non-negative $u \in H_0^1(\Omega)$, we have $u^* \in H_0^1(\Omega)$ and further

$$||u^*||_H \leq ||u||_H \tag{8.11}$$

where

$$||u||_H = \left\{ \int_\Omega |\nabla u|^2 d\mu \right\}^{\frac{1}{2}} \tag{8.12}$$

defines an equivalent norm on $H_0^1(\Omega)$.

8.3 Definitions

We recall the definition of a symmetric operator.

Definition Let $(\Omega, \mathcal{S}, \mu)$ be a measure space, let $1 \le p \le \infty$, and let q be the conjugate exponent of p.

A bounded linear operator $K : L^p(\Omega) \to L^q(\Omega)$ will be called *symmetric* if

$$\int_\Omega uKwd\mu = \int_\Omega wKud\mu \tag{8.13}$$

for all u and $w \in L^p(\Omega)$.

8.4 Generalised Solutions

In this section, we use standard methods to establish properties of the inverse Laplacian K, in a form relevant to the present context.

We say $u \in W^{1,2}(\Omega)$ satisfies $-\Delta u = 0$ (≥ 0, ≤ 0) respectively in the *generalised* (or *weak*) sense in Ω if

$$H(u, \varphi) = \int_\Omega \nabla u . \nabla \varphi = 0 \ (\le 0, \ge 0) \tag{8.14}$$

for all non-negative $\varphi \in C_0^1(\Omega)$. (The functions $\varphi \in C_0^1(\Omega)$ in the above definition are often referred to as *test functions.*)

For $u \in W^{1,2}(\Omega)$, we say that u satisfies $u \le 0$ on $\partial\Omega$ in the *weak* sense if its positive part $u^+ \in H_0^1(\Omega)$. Similarly $u \ge 0$ on $\partial\Omega$ in the *weak* sense if

$u^- \in H_0^1(\Omega)$. We say $u = 0$ on $\partial\Omega$ in the *weak* sense if $u \geq 0$ and $u \leq 0$ on $\partial\Omega$ in the weak sense. Other definitions of inequality at $\partial\Omega$ follow from the above. For two functions $u, v \in W^{1,2}(\Omega)$, we say $u \leq v$ in the *weak* sense if $u - v \leq 0$ in the weak sense i.e. $(u - v)_+ \in W_0^{1,2}(\Omega)$. We define

$$\sup_{\partial\Omega} u = \inf \{k | u \leq k \text{ on } \partial\Omega \text{ in the weak sense }, k \in \mathbf{R}\} \tag{8.15}$$

and

$$\inf_{\partial\Omega} u = -\sup_{\partial\Omega}(-u). \tag{8.16}$$

8.5 Existence of maximisers and their properties

Lemma 1 *Kv is well-defined for non-negative, non-zero $v \in L^p(\Omega)$, where $1 < p \leq 2$. Further, $K : L^p(\Omega) \to H_0^1(\Omega)$ is bounded and linear.*

Proof Let v be as above, and let q denote the conjugate exponent of p. We can characterise Kv as the unique minimiser over $u \in H_0^1(\Omega)$ of the convex functional defined by

$$\frac{1}{2}\int_\Omega |\nabla u|^2 d\mu - \int_\Omega uv d\mu. \tag{8.17}$$

By considering $u = 0$, we see that

$$\inf_{u \in H_0^1(\Omega)} \frac{1}{2}\int_\Omega |\nabla u|^2 d\mu - \int_\Omega uv d\mu \leq 0. \tag{8.18}$$

Let $Kv = u$. Then by (8.18),

$$\frac{1}{2}\int_\Omega |\nabla u|^2 d\mu \leq \int_\Omega uv d\mu \tag{8.19}$$

whence

$$\frac{1}{2}||u||_H^2 \leq \int_\Omega uv d\mu \leq ||u||_q ||v||_p \leq C_{1,2}(\Omega)||u||_H ||v||_p \tag{8.20}$$

where the latter inequality holds by the Sobolev Embedding Theorem.

Rewriting

$$||Kv||_H \leq 2C_{1,2}(\Omega)||v||_p. \tag{8.21}$$

Thus K is bounded. It is easy to see that K is linear. This completes the proof.

Lemma 2 $K : L^p(\Omega) \to H_0^1(\Omega)$ *is a positive (with respect to the usual order on function spaces) symmetric operator for p satisfying* $1 < p \leq 2$.

Proof Let $v \in L^p(\Omega)$, p as above, v non–negative. Then, by the definition of K, we have

$$-\Delta K v = v \text{ on } \Omega \tag{8.22}$$

$$Kv = 0 \text{ on } \partial\Omega \tag{8.23}$$

in the weak sense, as described in Section 8.4 (noting that $Kv \in H_0^1(\Omega)$). The generalised weak maximum principle stated in Chapter 7, Theorem 2 yields that $Kv \geq 0$. Thus K is a positive operator.

We show K is a symmetric operator. Let non–negative $v_1, v_2 \in L^p(\Omega)$, p as above. By Lemma 1, there exists $u_1, u_2 \in H_0^1(\Omega)$ such that $-\Delta u_1 = v_1$ and $-\Delta u_2 = v_2$ in the distributional sense.

We show that

$$\int_\Omega u_1 \Delta u_2 d\mu = -\int_\Omega \nabla u_1 . \nabla u_2 d\mu \tag{8.24}$$

Let $\{\phi_n\}_{n=1}^\infty \subset C_0^\infty(\Omega)$ be a sequence of test functions such that $\phi_n \to u_1$ in $H_0^1(\Omega)$. Then $\nabla\phi_n \to \nabla u_1$ in $L^2(\Omega)$. Moreover, by the Sobolev Embedding Theorem [**1**, Theorem 5.4B, page 97] we have $\phi_n \to u_1$ in $L^q(\Omega)$ where q denotes the conjugate exponent of p. Note that $u_2 \in H_0^1(\Omega)$, thus $\nabla u_2 \in L^2(\Omega)$. We have

$\phi_n \to u_1$ in $L^q(\Omega)$ and $\nabla\phi_n \to \nabla u_1$ in $L^2(\Omega)$, therefore

$$\begin{aligned}\int_\Omega u_1 \Delta u_2 d\mu &= \lim_{n\to\infty}\int_\Omega \phi_n \Delta u_2 d\mu && (8.25)\\ &= \lim_{n\to\infty} -\int_\Omega \nabla\phi_n.\nabla u_2 d\mu && (8.26)\\ &= -\int_\Omega \nabla u_1.\nabla u_2 d\mu. && (8.27)\end{aligned}$$

This verifies (8.24).

Now

$$\int_\Omega u_1 \Delta u_2 d\mu = \int_\Omega u_2 \Delta u_1 d\mu \tag{8.28}$$

where we have used equation (8.24). Rewriting yields

$$\int_\Omega v_1 K v_2 d\mu = \int_\Omega v_2 K v_1 d\mu \tag{8.29}$$

whence K is a symmetric operator. This completes the proof.

Lemma 3 *For $\lambda > 0$, define*

$$\varphi_\lambda(\xi) = \frac{1}{2}\int_\Omega \xi K \xi d\mu - \lambda \int_\Omega \xi y d\mu \tag{8.30}$$

where non–negative $\xi \in L^p(\Omega)$, for $1 < p \le 2$.

Let ξ^* denote the Steiner Symmetrisation (with respect to the y–axis) of ξ. Then

$$\varphi_\lambda(\xi^*) \ge \varphi_\lambda(\xi). \tag{8.31}$$

Proof For non–negative $w \in L^p(\Omega)$, p as above, we first show that

$$\int_\Omega w^* K w^* d\mu \ge \int_\Omega w K w d\mu. \tag{8.32}$$

From Lemma 1, we know $Kw \in H_0^1(\Omega)$. Using the methods of the proof of

Lemma 2, we can show

$$\int_{\Omega} \nabla Kw \nabla Kw d\mu = -\int_{\Omega} Kw \Delta Kw d\mu. \tag{8.33}$$

Now

$$\begin{aligned} ||Kw||_H^2 &= \int_{\Omega} |\nabla Kw|^2 d\mu && (8.34) \\ &= -\int_{\Omega} Kw \Delta Kw d\mu && (8.35) \\ &= \int_{\Omega} wKw d\mu. && (8.36) \end{aligned}$$

We proceed using the methods of Burton [**9**, Lemma 2]. Kw is defined to be the unique minimiser over $h \in H_0^1(\Omega)$ of the functional

$$\frac{1}{2}\int_{\Omega} |\nabla h|^2 d\mu - \int_{\Omega} hw d\mu. \tag{8.37}$$

Hence we have

$$-\frac{1}{2}\int_{\Omega} wKw d\mu = \inf_{h \in H_0^1(\Omega)} \left\{ \frac{1}{2}||h||_H^2 - \int_{\Omega} hw d\mu \right\}. \tag{8.38}$$

We write $Kw = u$. By (8.38) we have

$$-\frac{1}{2}\int_{\Omega} wKw d\mu = \frac{1}{2}||u||_H^2 - \int_{\Omega} uw d\mu. \tag{8.39}$$

By methods analogous to those used to obtain (8.38), we have

$$-\frac{1}{2}\int_{\Omega} w^* Kw^* d\mu = \inf_{h \in H_0^1(\Omega)} \left\{ \frac{1}{2}||h||_H^2 - \int_{\Omega} hw^* d\mu \right\}. \tag{8.40}$$

From (8.40) we obtain

$$-\frac{1}{2}\int_{\Omega} w^* K w^* d\mu \leq \frac{1}{2}||u^*||_H^2 - \int_{\Omega} u^* w^* d\mu. \tag{8.41}$$

Now from (8.39) and (8.41),

$$\begin{aligned} & \frac{1}{2}\int_{\Omega} w^* K w^* d\mu - \frac{1}{2}\int_{\Omega} wKw d\mu & (8.42) \\ \geq & \int_{\Omega} u^* w^* d\mu - \frac{1}{2}||u^*||_H^2 + \frac{1}{2}||u||_H^2 - \int_{\Omega} uw d\mu \geq 0. & (8.43) \end{aligned}$$

using the Steiner symmetrisation inequalities (8.10) and (8.11) (noting that u is non–negative by Lemma 2).

Now for non–negative $w \in L^p(\Omega)$, wy and w^*y are μ–rearrangements. Thus

$$\int_{\Omega} wy d\mu = \int_{\Omega} w^* y d\mu \tag{8.44}$$

(in the sense that if either integral is finite, then so is the other, and they are equal). Combining the above, for non–negative $\xi \in L^p(\Omega)$, $1 < p \leq 2$,

$$\begin{aligned} \varphi_\lambda(\xi^*) &= \frac{1}{2}\int_{\Omega} \xi^* K \xi^* d\mu - \lambda \int_{\Omega} \xi^* y d\mu & (8.45) \\ &\geq \frac{1}{2}\int_{\Omega} \xi K \xi d\mu - \lambda \int_{\Omega} \xi y d\mu = \varphi_\lambda(\xi). & (8.46) \end{aligned}$$

This completes the proof.

Corollary 1 *For $\lambda > 0$, $\varphi_\lambda : (L^p(\Omega))^+ \to \mathbf{R}$ is a strictly convex function, for $1 < p \leq 2$. ($(L^p(\Omega))^+$ denotes the non–negative $L^p(\Omega)$ functions.)*

Proof Let non–negative $\xi \in L^p(\Omega)$, for $1 < p \leq 2$. Then

$$\int_{\Omega} \xi K \xi d\mu = ||K\xi||_H^2 \tag{8.47}$$

using the methods of the proof of Lemma 3. Thus $\xi \to \int_\Omega \xi K\xi$ is a strictly convex map by strict convexity of $||.||_H^2$ and linearity and injectivity of K. The map $\xi \to \lambda \int_\Omega \xi y d\mu$ is linear, whence φ_λ is strictly convex. This completes the proof.

Lemma 4 *Let $S(\Omega)_p^+$ denote the set of all non–negative Steiner symmetric functions in $L^p(\Omega)$. That is,*

$$S(\Omega)_p^+ = \{v \geq 0 | v \in L^p(\Omega), v \text{ symmetrically decreasing in } x \text{ for a.e. } y \in (0,R)\} \tag{8.48}$$

Consider $T : S(\Omega)_p^+ \to \overline{\mathbf{R}}$, where

$$T(\xi) = \int_\Omega \xi K\xi d\mu \tag{8.49}$$

and $1 < p < 2$. Then T is weakly sequentially continuous.

Proof We show that

$$K(S(\Omega)_p^+) \subset \{w \geq 0,\ w \in H_0^1(\Omega) | w \text{ symmetrically decreasing in } x\} \tag{8.50}$$

Applying the results of Lemmas 1 and 2 we know that $K : L^p(\Omega) \to H_0^1(\Omega)$ is a positive bounded map. It remains to show that for non–negative $\xi \in L^p(\Omega)$, p as above, ξ symmetrically decreasing in x, we have $K\xi$ symmetrically decreasing in x. Let ξ be as above. Note that $\xi^* = \xi$. $K\xi^*$ is defined to be the unique minimiser over $u \in H_0^1(\Omega)$ of the functional

$$\frac{1}{2}\int_\Omega |\nabla u|^2 d\mu - \int_\Omega u\xi^* d\mu \equiv \Phi(u), \text{ say.} \tag{8.51}$$

Let $K\xi^* = u$. Then

$$\Phi(u^*) = \frac{1}{2}\int_\Omega |\nabla u^*|^2 d\mu - \int_\Omega u^* \xi^* d\mu \tag{8.52}$$
$$\leq \frac{1}{2}\int_\Omega |\nabla u|^2 d\mu - \int_\Omega u \xi^* d\mu \tag{8.53}$$

using inequalities (8.10) and (8.11) from Section 8.2. Thus

$$\Phi(u^*) \leq \Phi(u). \tag{8.54}$$

Since u is the unique minimiser we have $u = u^*$, that is Kv^* is symmetrically decreasing in x.

By [**24**, Theorem III.1] the set

$$\{u \geq 0 | u \in H_0^1(\Omega), u \text{ symmetrically decreasing in } x\} \tag{8.55}$$

is compactly embedded in $L^q(\Omega)$ where q denotes the conjugate exponent of p. (This result is valid for $2 < q < \infty$). Combining the above results, the map $K : S(\Omega)_p^+ \to L^q(\Omega)$ is compact.

We now show that T is weakly sequentially continuous. Let $u_n \xrightarrow{w} u_0$, where $\{u_n\}_{n=1}^\infty \subset S(\Omega)_p^+$, and $u_0 \in S(\Omega)_p^+$. Then

$$\left| \int_\Omega (u_0 - u_n)(Ku_0 - Ku_n) d\mu \right| \leq ||u_0 - u_n||_p ||Ku_0 - Ku_n||_q \tag{8.56}$$

and the right hand side tends to 0 as $n \to \infty$ because K is compact. Thus

$$\lim_{n\to\infty} \int_\Omega (u_n - u_0)(Ku_n - Ku_0) d\mu = 0. \tag{8.57}$$

Rewriting we obtain

$$\lim_{n\to\infty}\left\{\int_\Omega u_0 K u_0 d\mu + \int_\Omega u_n K u_n d\mu - \int_\Omega u_n K u_0 d\mu - \int_\Omega u_0 K u_n d\mu\right\} = 0 \quad (8.58)$$

whence

$$\int_\Omega u_0 K u_0 d\mu + \lim_{n\to\infty}\int_\Omega u_n K u_n d\mu - \int_\Omega u_0 K u_0 d\mu - \int_\Omega u_0 K u_0 d\mu = 0 \quad (8.59)$$

therefore

$$\lim_{n\to\infty}\int_\Omega u_n K u_n d\mu = \int_\Omega u_0 K u_0 d\mu. \quad (8.60)$$

Thus T is weakly sequentially continuous on $S(\Omega)_p^+$. This completes the proof.

Lemma 5 *Let non–negative $\xi \in L^p(\Omega)$, $1 \le p < \infty$. Then the map*

$$\xi \to \int_\Omega \xi y d\mu \quad (8.61)$$

is weakly lower semicontinuous.

Proof Define $f : \Omega \times \mathbf{R} \to \overline{\mathbf{R}}$,

$$f((x,y),z) = \begin{cases} yz & \text{if } z \ge 0 \\ \infty & \text{if } z < 0 \end{cases}$$

Then f is a non–negative normal integrand. Now let $\xi \in L^p(\Omega)$, $1 \le p < \infty$. Define

$$F(\xi) = \int_\Omega f((x,y),\xi(x,y))d\mu. \quad (8.62)$$

[**15**, Corollary 1.2, page 239] yields that F is a lower semicontinuous function. Furthermore since $f((x,y),.)$ is convex for almost every $(x,y) \in \Omega$, F defines a convex function on $L^p(\Omega)$, p as above. Combining the above results, we obtain the fact that F defines a weakly lower semicontinuous function on $L^p(\Omega)$, p as above. It is elementary to show the restriction of F to non–negative $L^p(\Omega)$ functions is weakly lower semicontinuous.

$F : (L^p(\Omega))^+ \to \overline{\mathbb{R}}$ may be written

$$F(u) = \int_\Omega u y d\mu. \tag{8.63}$$

This completes the proof.

Theorem 1 *Let non-negative, non-zero $f_0 \in L^1(\Omega) \bigcap L^p(\Omega)$, $1 < p < 2$. For every $\lambda > 0$,*

(i) φ_λ attains a maximum relative to $\overline{R(f_0)}^w$.

(ii) All maximisers in (i) belong to $RC(f_0)$.

Proof (i) Recall, for $\lambda > 0$, $\xi \in \overline{R(f_0)}^w$,

$$\varphi_\lambda(\xi) = \frac{1}{2} \int_\Omega \xi K \xi d\mu - \lambda \int_\Omega \xi y d\mu. \tag{8.64}$$

Let $\{\xi_n\}_{n=1}^\infty$ be a maximising sequence in $\overline{R(f_0)}^w$ for φ_λ. Replace ξ_n by ξ_n^*, the Steiner Symmetrisation of ξ_n with respect to the y–axis. By Lemma 3, $\varphi_\lambda(\xi_n^*) \geq \varphi_\lambda(\xi_n)$ $\forall n \in \mathbb{N}$, therefore $\{\xi_n^*\}_{n=1}^\infty$ is a maximising sequence relative to $\overline{R(f_0)}^w$ for φ_λ.

Let $\overline{R}^*$ denote the set of elements v of $\overline{R(f_0)}^w$ such that $v = v^*$. $\overline{R}^*$ is immediately seen to be convex, and the Steiner Symmetrisation inequality (8.9) in Section 8.2 yields that it is closed. Thus $\overline{R}^*$ is weakly closed.

$\overline{R(f_0)}^w$ is weakly compact by Chapter 3, Theorem 2, thus passing to a subsequence if necessary, $\xi_n^* \xrightarrow{w} \xi_0 \in \overline{R}^*$ (because $\overline{R}^*$ is weakly closed).

Appealing to Lemmas 4 and 5 we obtain

$$\begin{aligned} \varphi_\lambda(\xi_0) &= \frac{1}{2} \int_\Omega \xi_0 K \xi_0 d\mu - \lambda \int_\Omega \xi_0 y d\mu && (8.65) \\ &\geq \lim_{n\to\infty} \frac{1}{2} \int_\Omega \xi_n^* K \xi_n^* d\mu - \lambda \liminf_{n\to\infty} \int_\Omega \xi_n^* y d\mu && (8.66) \\ &= \limsup_{n\to\infty} \frac{1}{2} \int_\Omega \xi_n^* K \xi_n^* d\mu + \limsup_{n\to\infty} \left\{ -\lambda \int_\Omega \xi_n^* y d\mu \right\} && (8.67) \end{aligned}$$

$$\geq \limsup_{n\to\infty} \varphi_\lambda(\xi_n^*) \geq \sup_{\xi\in\overline{R(f_0)}^w} \varphi_\lambda(\xi). \tag{8.68}$$

Thus φ_λ attains a maximum relative to $\overline{R(f_0)}^w$.

(ii) For $\lambda > 0$, φ_λ is a strictly convex function. (This follows from Corollary 1.) Let $\xi_0 \in \overline{R(f_0)}^w$ be a maximiser for φ_λ relative to $\overline{R(f_0)}^w$. Then ξ_0 is an extreme point of $\overline{R(f_0)}^w$, (by strict convexity of φ_λ and Lemma 6 of Chapter 6), therefore by Chapter 3, Theorem 2 we have $\xi_0 \in RC(f_0)$. This completes the proof.

Lemma 6 *Let non-negative non-zero* $v \in L^1(\Omega)\bigcap L^2(\Omega)$. *Then* $Kv - \lambda y \in L^\infty(\Omega)$, *for* $\lambda > 0$.

Proof It is immediate that $\lambda y \in L^\infty(\Omega)$, thus it remains to show $Kv \in L^\infty(\Omega)$. For v as above, we recall the definition of the operator K. Kv is defined as the unique minimiser over $u \in H_0^1(\Omega)$ of the functional

$$\varphi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 d\mu - \int_\Omega uv d\mu. \tag{8.69}$$

Now Kv satisfies

$$-\triangle Kv = v \text{ on } \Omega \tag{8.70}$$

$$Kv = 0 \text{ on } \partial\Omega \tag{8.71}$$

where $Kv = 0$ in the weak sense. Let P be the upper half–plane, that is $P = (-\infty,\infty)\times(0,\infty)$. For non–negative $u \in L^1(P)\bigcap L^2(P)$, we define the operator $\tilde{K}$ by

$$\tilde{K}v(z) = \int_P G(z,w)v(w)dw \tag{8.72}$$

where

$$G(z,w) = \frac{1}{2\pi}\log\frac{|z-\bar{w}|}{|z-w|} \tag{8.73}$$

for $z, w \in P$, where $\bar{w}$ denotes the reflection of w in the x–axis, and $|.|$ denotes the Euclidean distance in $\mathbf{R}^2$. We aim to show that $Kv(z) \leq \tilde{K}v(z)$ for almost every $z \in \Omega$. For $n \in \mathbf{N}$, we define $\Omega_n = (-n, n) \times (0, R)$, a bounded subdomain of Ω. We define $K_n(v)$ as the unique minimiser over $u \in H_0^1(\Omega_n)$ of the functional

$$\varphi_n(u) = \frac{1}{2}\int_{\Omega_n} |\nabla u|^2 d\mu - \int_{\Omega_n} uv d\mu \tag{8.74}$$

The following equations are satisfied

$$-\Delta K_n v = v \text{ on } \Omega_n \tag{8.75}$$

$$K_n v = 0 \text{ on } \partial\Omega_n \tag{8.76}$$

where $K_n v = 0$ on $\partial\Omega_n$ in the weak sense (because $K_n v \in H_0^1(\Omega_n)$ by the definition of the operator K_n).

For each $n \in \mathbf{N}$, a special case of the Calderon–Zygmund inequality [**20**, Theorem 9.9, page 230] yields that $\tilde{K}v_n \in W^{2,2}(\Omega_n)$, where v_n denotes $v|_{\Omega_n}$. By way of explanation, we note that $\tilde{K}v_n$ is the difference of two Newtonian potentials. From equation (8.72) and [**20**, Theorem 9.9, page 230] we obtain

$$-\Delta\tilde{K}v_n = v \text{ on } \Omega_n \tag{8.77}$$

$$\tilde{K}v_n \geq 0 \text{ on } \partial\Omega_n \tag{8.78}$$

where equation (8.77) holds almost everywhere, and $\tilde{K}v_n \geq 0$ on $\partial\Omega_n$ in the weak sense (since $\tilde{K}v_n(z) \geq 0$ in the usual sense for all $z \in \Omega_n$ by definition). Combining (8.75)–(8.76) and (8.77)–(8.78) we obtain, for a given $n \in \mathbf{N}$,

$$-\Delta(\tilde{K}v_n - K_n v) = 0 \text{ on } \Omega_n \tag{8.79}$$

$$\tilde{K}v_n - K_n v \geq 0 \text{ on } \partial\Omega_n \tag{8.80}$$

where $\tilde{K}v_n - K_n v \geq 0$ on $\partial\Omega_n$ in the weak sense. By way of explanation we note that $\tilde{K}v_n \in W^{1,2}(\Omega_n)$, $(\tilde{K}v_n)^- \in H_0^1(\Omega_n)$ and $K_n v \in H_0^1(\Omega_n)$, therefore we have $(\tilde{K}v_n - K_n v)^- \in H_0^1(\Omega_n)$, (we show this in the Appendix) which is exactly the statement that $\tilde{K}v_n - K_n v \geq 0$ on $\partial\Omega_n$ in the weak sense. Our regularity results yield that $\tilde{K}v_n - K_n v \in W^{1,2}(\Omega_n)$, and applying a generalised weak maximum principle [**20**, Theorem 8.1, page 179] we obtain that $\inf_{\Omega_n}(\tilde{K}v_n - K_n v) \geq \inf_{\partial\Omega_n}(\tilde{K}v_n - K_n v)^- = 0$, (where we mean $\inf_{\partial\Omega}(\tilde{K}v_n - K_n v)^-$ in the sense described in Section 8.4) whence

$$\tilde{K}v_n(z) \geq K_n v(z) \text{ for almost every } z \in \Omega_n \tag{8.81}$$

for every $n \in \mathbf{N}$, whence

$$\tilde{K}v(z) \geq K_n v(z) \tag{8.82}$$

for almost every $z \in \Omega_n$, for every $n \in \mathbf{N}$.

We show that $\{K_n v\}_{n=1}^{\infty}$ is a minimising sequence for φ over $H_0^1(\Omega)$. Let $\epsilon > 0$ be given. Define

$$E_\epsilon = \left\{ u \in H_0^1(\Omega) | \varphi(u) < \inf_{u \in H_0^1(\Omega)} \varphi(u) + \epsilon \right\}. \tag{8.83}$$

Now E_ϵ is non–empty (by the definition of infimum) and open (by the continuity of φ). $C_0^\infty(\Omega)$ functions are dense in $H_0^1(\Omega)$, therefore there exists $\theta_n \in C_0^\infty(\Omega) \bigcap E_\epsilon$ where supp $\theta_n \subset \Omega_n$ for some $n \in \mathbf{N}$. Now $\inf_{u \in H_0^1(\Omega_n)} \varphi(u)$ is a decreasing function of n, therefore

$$\varphi(K_m v) \leq \varphi(\theta_n) < \inf_{u \in H_0^1(\Omega)} \varphi(u) + \epsilon \text{ for all } m \geq n. \tag{8.84}$$

Thus $\{K_n v\}_{n=1}^{\infty}$ is a minimising sequence for φ, that is

$$\varphi(K_n v) \to \inf_{u \in H_0^1(\Omega)} \varphi(u) = \varphi(Kv) \text{ as } n \to \infty. \tag{8.85}$$

Using the methods of a previous lemma, noting that φ is coercive, strictly convex and weakly lower semicontinuous, passing to a subsequence if necessary we obtain

$$K_n v \xrightarrow{w} Kv \text{ as } n \to \infty. \tag{8.86}$$

A previous lemma yields that

$$\int_{\Omega} vKv d\mu = ||Kv||_H^2 \tag{8.87}$$

and it may be shown that

$$\int_{\Omega_n} vK_n v d\mu = ||K_n v||_{H,\Omega_n}^2 = ||K_n v||_H^2 \tag{8.88}$$

for v as above. Therefore (8.85) yields that

$$||K_n v||_H \to ||Kv||_H \text{ as } n \to \infty. \tag{8.89}$$

Combining (8.86) and (8.89) we obtain $K_n v \to Kv$ in $H_0^1(\Omega)$. By the Corollary to the Riesz–Fischer Theorem, $K_n v \to Kv$ pointwise almost everywhere. Recalling (8.82),

$$K_n v(z) \leq \tilde{K}v(z), \text{ for all } z \in \Omega_n, \text{ for all } n \in \mathbb{N} \tag{8.90}$$

we have

$$Kv(z) \leq \tilde{K}v(z) \text{ for } z \in \Omega \tag{8.91}$$

as required. Now

$$\tilde{K}v(z) = \int_P G(z,w)v(w)dw = \int_\Omega G(z,w)v(w)dw. \tag{8.92}$$

We change to polar co–ordinates about z. Let $\rho = |z-w|$. Then by (8.91),

$$Kv(z) \leq \int_{\{w\in\Omega|\rho\geq 1\}} G(z,w)v(w)dw + \int_{\{w\in\Omega|\rho<1\}} G(z,w)v(w)dw. \tag{8.93}$$

For $\rho \geq 1$, we have

$$\rho \leq |z-\bar{w}| \leq |z-w| + |w-\bar{w}| \leq \rho + 2R \leq (2R+1)\rho \tag{8.94}$$

whence

$$(2R+1) \geq \frac{|z-\bar{w}|}{|z-w|} \geq 1 \text{ for } w, z \in \Omega, \text{ s.t. } |w-z| \geq 1. \tag{8.95}$$

Therefore

$$\log(2R+1) \geq \log\frac{|z-\bar{w}|}{|z-w|} \geq 0 \text{ for } w, z \in \Omega, \text{ s.t. } |w-z| \geq 1. \tag{8.96}$$

Thus we have

$$\left|\int_{\{w\in\Omega|\rho\geq 1\}} G(z,w)v(w)dw\right| \leq \frac{1}{2\pi}\log(2R+1)||v||_1. \tag{8.97}$$

For $\rho < 1$, we have

$$\rho \leq |z-\bar{w}| \leq |z-w| + |w-\bar{w}| < 1 + 2R \tag{8.98}$$

whence

$$1 \leq \frac{|z-\bar{w}|}{|z-w|} < \frac{(1+2R)}{\rho} \text{ for } w, z \in \Omega, \text{ s.t. } |w-z| < 1. \tag{8.99}$$

Thus we have

$$
\begin{aligned}
& \int_{\{w\in\Omega|\rho<1\}} G(z,w)v(w)dw && (8.100)\\
< \; & \frac{1}{2\pi}\log(1+2R)||v||_1 - \frac{1}{2\pi}\int_{\{w\in\Omega|\rho<1\}} v(w)\log\rho dw && (8.101)\\
= \; & \frac{1}{2\pi}\log(1+2R)||v||_1 - \int_0^{2\pi}\int_0^1 \frac{1}{2\pi} v\log\rho\rho d\rho d\theta && (8.102)\\
\leq \; & \frac{1}{2\pi}\log(1+2R)||v||_1 + ||v||_p\left\{\int_0^1 |\log\rho|^q\rho d\rho\right\}^{\frac{1}{q}} && (8.103)\\
\leq \; & \frac{1}{2\pi}\log(1+2R)||v||_1 + \kappa||v||_p. && (8.104)
\end{aligned}
$$

where q denotes the conjugate exponent of p, and for κ some constant, since $\int_0^1 |(\log\rho)|^q\rho d\rho$ is finite (this is verified in the Appendix). Combining (8.91) and (8.97),(8.100)–(8.104) we obtain that $Kv \in L^\infty(\Omega)$. This completes the proof.

Lemma 7 *Let non–negative $f_0 \in L^1(\Omega)\bigcap L^2(\Omega)$. Suppose, for a given $\lambda > 0$, φ_λ has a maximiser $\tilde{f}$ relative to $R(f_0)$. Then $\tilde{u} = K\tilde{f}$ satisfies $-\triangle\tilde{u} = \phi(\tilde{u} - \lambda y)$, in the weak sense, for some increasing function ϕ.*

Proof Recall, for $\lambda > 0$ and non–negative $v \in L^1(\Omega)\bigcap L^2(\Omega)$,

$$\varphi_\lambda(v) = \frac{1}{2}\int_\Omega vKvd\mu - \lambda\int_\Omega vyd\mu. \tag{8.105}$$

Let $h \in L^1(\Omega)\bigcap L^2(\Omega)$ (we work in the space $L^1(\Omega)\bigcap L^2(\Omega)$ with the norm $||.|| = ||.||_1 + ||.||_2$) , then

$$
\begin{aligned}
& \varphi_\lambda(v+h) - \varphi_\lambda(v) && (8.106)\\
= \; & \frac{1}{2}\int_\Omega (v+h)K(v+h)d\mu - \lambda\int_\Omega (v+h)yd\mu \\
& -\frac{1}{2}\int_\Omega vKvd\mu + \lambda\int_\Omega vyd\mu && (8.107)\\
= \; & \int_\Omega hKvd\mu + \frac{1}{2}\int_\Omega hKhd\mu - \lambda\int_\Omega hyd\mu && (8.108)
\end{aligned}
$$

where we have used the fact that K is a symmetric operator (shown in Lemma 2). Consequently the Fréchet derivative of φ_λ, which we will denote $d\varphi_\lambda[v]$, is given by

$$d\varphi_\lambda[v] = Kv - \lambda y. \tag{8.109}$$

By way of explanation, we note that $d\varphi_\lambda[v] \in L^\infty(\Omega)$, by Lemma 6. Therefore $d\varphi_\lambda[v]$ belongs to the dual of the space $L^1(\Omega) \bigcap L^2(\Omega)$. φ_λ is a strictly convex and Fréchet differentiable function, thus φ_λ is sub–differentiable at v, for v as above, and $\partial\varphi_\lambda(v) = \{d\varphi_\lambda[v]\}$. Therefore, for every $f \in R(f_0)\backslash\{\tilde{f}\}$ we have

$$\varphi_\lambda(\tilde{f}) \geq \varphi_\lambda(f) > \int_\Omega (K\tilde{f} - \lambda y)(f - \tilde{f})d\mu + \varphi_\lambda(\tilde{f}) \tag{8.110}$$

where the strict inequality follows by the strict convexity of φ_λ . Rearranging, we obtain

$$\int_\Omega (K\tilde{f} - \lambda y) f d\mu < \int_\Omega (K\tilde{f} - \lambda y)\tilde{f} d\mu, \ \forall f \in R(f_0)\backslash\{\tilde{f}\}. \tag{8.111}$$

We now apply Theorem 1, Chapter 7 to $\tilde{f} \in L^1(\Omega)$ and $K\tilde{f} - \lambda y \in L^\infty(\Omega)$ to obtain

$$\tilde{f} = \phi(K\tilde{f} - \lambda y) \tag{8.112}$$

for some increasing function ϕ, μ almost everywhere. Now $K\tilde{f} = \tilde{u}$ for some $\tilde{u} \in H_0^1(\Omega)$. Accordingly, we obtain

$$-\Delta\tilde{u} = \phi(\tilde{u} - \lambda y) \tag{8.113}$$

in the weak sense. This completes the proof.

Lemma 8 *Let non–negative* $f_0 \in L^1(\Omega) \bigcap L^2(\Omega)$. *Let* $\lambda > 0$. *Let* $\tilde{f}$ *be a max-*

imiser for φ_λ relative to $\overline{R(f_0)}^w$. (We know such a maximiser exists). Then

$$\tilde{f}^{-1}(0,\infty) \subset (K\tilde{f} - \lambda y)^{-1}(0,\infty) \tag{8.114}$$

except possibly for a set of measure zero, whence

$$\mu(\tilde{f}^{-1}(0,\infty)) \leq \mu((K\tilde{f} - \lambda y)^{-1}(0,\infty)) \tag{8.115}$$

(when the above expression makes sense)

Proof The proof of Lemma 7 yields that φ_λ is sub–differentiable at $\tilde{f}$, and that $\partial\varphi_\lambda(\tilde{f}) = \{K\tilde{f} - \lambda y\}$. Therefore, for every $f \in \overline{R(f_0)}^w \backslash \{\tilde{f}\}$ we have

$$\varphi_\lambda(\tilde{f}) \geq \varphi_\lambda(f) > \int_\Omega (K\tilde{f} - \lambda y)(f - \tilde{f})d\mu + \varphi_\lambda(\tilde{f}) \tag{8.116}$$

where the strict inequality follows by the strict convexity of φ_λ. Rearranging, we obtain

$$\int_\Omega (K\tilde{f} - \lambda y)f d\mu < \int_\Omega (K\tilde{f} - \lambda y)\tilde{f} d\mu, \ \forall f \in \overline{R(f_0)}^w \backslash \{\tilde{f}\}. \tag{8.117}$$

Let $S = \tilde{f}^{-1}(0,\infty)$. We suppose the result is false, for a contradiction. Then there exists $A \subset S$, a set of positive measure such that $K\tilde{f} - \lambda y \leq 0$ on A. Define

$$\bar{f}(z) = \begin{cases} 0 & z \in A \\ \tilde{f}(z) & z \in \Omega \backslash A \end{cases}$$

for $z \in \Omega$. Then $\bar{f} \in \overline{R(f_0)}^w$ (by Chapter 3, Theorem 2, which characterises the weak closure of the set of rearrangements) , $\bar{f} \neq \tilde{f}$, and

$$\int_\Omega (K\tilde{f} - \lambda y)\bar{f} d\mu \geq \int_\Omega (K\tilde{f} - \lambda y)\tilde{f} d\mu \tag{8.118}$$

which contradicts (8.117). This establishes (8.114), and (8.115) follows. This completes the proof.

Lemma 9 *Let non-negative $f_0 \in L^1(\Omega) \bigcap L^2(\Omega)$. Fix $\lambda > 0$. Let $\tilde{f}$ be a maximiser for φ_λ relative to $\overline{R(f_0)}^w$. Suppose that $\tilde{f} \notin R(f_0)$. Then*

$$(K\tilde{f} - \lambda y)^{-1}(0, \infty) = \tilde{f}^{-1}(0, \infty) \tag{8.119}$$

except possibly for a set of measure zero.

Proof In Lemma 8, it was shown that

$$\tilde{f}^{-1}(0, \infty) \subset (K\tilde{f} - \lambda y)^{-1}(0, \infty) \tag{8.120}$$

except possibly for a set of measure zero. We show that

$$(K\tilde{f} - \lambda y)^{-1}(0, \infty) \subset \tilde{f}^{-1}(0, \infty) \tag{8.121}$$

except possibly for a set of measure zero. We suppose the statement is false, to show a contradiction. Therefore there exists $V^1 \subset (K\tilde{f} - \lambda y)^{-1}(0, \infty)$, a set of positive measure, such that $V^1 \subset \tilde{f}^{-1}(0)$. We have previously seen that $\tilde{f} \in RC(f_0)$, and by assumption $\tilde{f} \notin R(f_0)$. The definition of $RC(f_0) \backslash R(f_0)$ shows that there exists $\alpha > \beta > 0$ such that $\nu\left(f_0^{-1}(\beta, \alpha]\right) > \nu\left(\tilde{f}^{-1}(\beta, \alpha]\right)$. Let $\delta = \nu\left(f_0^{-1}(\beta, \alpha]\right) - \nu\left(\tilde{f}^{-1}(\beta, \alpha]\right)$. Let $V \subset V^1$, where $0 < \nu(V) \leq \delta$. (The existence of such a V is established by arguments similar to those used in the proof of Theorem 1 of Chapter 7). Define

$$\hat{f}(x) = \begin{cases} \tilde{f}(x) & \text{if } x \in \Omega \backslash V \\ \beta & \text{if } x \in V \end{cases}$$

Now $\hat{f}$ is non–negative, and $\hat{f} \neq \tilde{f}$. Moreover, for any positive real number σ

$$\int_{\Omega}(\hat{f} - \sigma)_+ d\mu \leq \int_{\Omega}(f_0 - \sigma)_+ d\mu. \tag{8.122}$$

The characterisation of the weak closure of the set of rearrangements given in Chapter 3, Theorem 2, yields that $\hat{f} \in \overline{R(f_0)}^w$. Recall, from the proof of Lemma 8, that

$$\int_{\Omega}(K\tilde{f} - \lambda y) f d\mu < \int_{\Omega}(K\tilde{f} - \lambda y)\tilde{f} d\mu \tag{8.123}$$

for every $f \in \overline{R(f_0)}^w \backslash \{\tilde{f}\}$. However $\hat{f} \in \overline{R(f_0)}^w$, and $\hat{f} \neq \tilde{f}$, and further

$$\begin{aligned}
\int_{\Omega}(K\tilde{f} - \lambda y)\tilde{f} d\mu &= \int_{\Omega\backslash V}(K\tilde{f} - \lambda y)\tilde{f} d\mu && (8.124)\\
&= \int_{\Omega\backslash V}(K\tilde{f} - \lambda y)\hat{f} d\mu && (8.125)\\
&< \int_{\Omega}(K\tilde{f} - \lambda y)\hat{f} d\mu. && (8.126)
\end{aligned}$$

This contradicts (8.123), thus establishing (8.121). This completes the proof.

Corollary 2 *Let non–negative* $f_0 \in L^1(\Omega)\bigcap L^2(\Omega)$. *Fix* $\lambda > 0$. *Let* $\tilde{f}$ *be a maximiser for* φ_λ *relative to* $\overline{R(f_0)}^w$. *Suppose that* $\tilde{f} \notin R(f_0)$. *Then* $(K\tilde{f} - \lambda y)^{-1}(0)$ *has zero measure.*

Proof For $f \in \overline{R(f_0)}^w \backslash \{\tilde{f}\}$ we have

$$\int_{\Omega}(K\tilde{f} - \lambda y) f d\mu < \int_{\Omega}(K\tilde{f} - \lambda y)\tilde{f} d\mu. \tag{8.127}$$

If $(K\tilde{f} - \lambda y)^{-1}(0)$ has positive measure, then the methods of the previous lemma yield the existence of $\hat{f} \in \overline{R(f_0)}^w$, $\hat{f} \neq \tilde{f}$, such that

$$\int_{\Omega}(K\tilde{f} - \lambda y)\tilde{f} d\mu = \int_{\Omega}(K\tilde{f} - \lambda y)\hat{f} d\mu \tag{8.128}$$

contradicting (8.127). This completes the proof.

Lemma 10 *Let non-negative non-zero $f_0 \in L^1(\Omega)\bigcap L^p(\Omega)$, for $p > 2$. Let $\tilde{f}$ be a maximiser for φ_λ relative to $\overline{R(f_0)}^w$. Then, for a given $\epsilon > 0$, there exists $M > 0$ such that for $|x| > M$,*

$$0 \leq K\tilde{f}(x,y) < \epsilon y \tag{8.129}$$

Proof For non-negative $v \in L^1(\Omega)\bigcap L^p(\Omega)$, for p as above, we recall the definition of $\tilde{K}$.

$$\tilde{K}v(z) = \int_\Omega G(z,w)v(w)dw \tag{8.130}$$

for $z, w \in \Omega$, where

$$G(z,w) = \frac{1}{2\pi}\log\frac{|z-\bar{w}|}{|z-w|} \tag{8.131}$$

where $\bar{w}$ denotes the reflection of w in the x-axis, and $|.|$ denotes the Euclidean distance in $\mathbf{R}^2$.

Let $z = (x_1, y_1)$, $w = (x_2, y_2)$. By the proof of Lemma 6 we have

$$\tilde{K}v(z) \geq Kv(z) \text{ for every } z \in \Omega. \tag{8.132}$$

Therefore, (noting that $G(x_1, 0, x_2, y_2) = 0$),

$$Kv(x_1,y_1) \leq \tilde{K}v(x_1,y_1) = \int_\Omega G(x_1,y_1,x_2,y_2)v(x_2,y_2)dx_2dy_2 \tag{8.133}$$

$$= \int_\Omega\int_0^{y_1}\frac{\partial G}{\partial y_1}(x_1,y_1,x_2,y_2)dy_1v(x_2,y_2)dx_2dy_2. \tag{8.134}$$

$\frac{\partial G}{\partial y_1}(x_1,y_1,x_2,y_2)v(x_2,y_2)$ is a non-negative measurable function with respect to $(y, x_2, y_2) \in (0, y_1) \times \Omega$. Applying Fubini's theorem

$$\tilde{K}v(x_1,y_1) = \int_0^{y_1}\left\{\int_\Omega\frac{\partial G}{\partial y_1}(x_1,y_1,x_2,y_2)v(x_2,y_2)dx_2dy_2\right\}dy_1. \tag{8.135}$$

To complete the proof it is sufficient to show the term in brackets tends to zero as $M \to \infty$, where $|x_1| > M$. Now

$$2\pi \frac{\partial G}{\partial y_1}(x_1, y_1, x_2, y_2) = \frac{y_1 + y_2}{(x_1 - x_2)^2 + (y_1 + y_2)^2} - \frac{y_1 - y_2}{(x_1 - x_2)^2 + (y_1 - y_2)^2} \tag{8.136}$$

$$\begin{aligned}
&\leq \frac{\{(x_1 - x_2)^2 + (y_1 + y_2)^2\}^{\frac{1}{2}}}{(x_1 - x_2)^2 + (y_1 + y_2)^2} + \frac{\{(x_1 - x_2)^2 + (y_1 - y_2)^2\}^{\frac{1}{2}}}{(x_1 - x_2)^2 + (y_1 - y_2)^2} && (8.137)\\
&= \left\{(x_1 - x_2)^2 + (y_1 + y_2)^2\right\}^{-\frac{1}{2}} + \left\{(x_1 - x_2)^2 + (y_1 - y_2)^2\right\}^{-\frac{1}{2}} && (8.138)\\
&\leq 2\left\{(x_1 - x_2)^2 + (y_1 - y_2)^2\right\}^{-\frac{1}{2}} && (8.139)
\end{aligned}$$

since $|y_1 - y_2| \leq y_1 + y_2$. Writing

$$\rho = \left\{(x_1 - x_2)^2 + (y_1 - y_2)^2\right\}^{\frac{1}{2}} \tag{8.140}$$

for $|x_1| > 2M$, (where $M > 1$), and letting $p \geq p_1 > 2$ and $1 < p_2 < 2$ (where q_1, q_2 denote the conjugate exponents of p_1, p_2 respectively), we have

$$\begin{aligned}
&\int_\Omega \frac{\partial G}{\partial y_1}(x_1, y_1, x_2, y_2) v(x_2, y_2) dx_2 dy_2 \leq \int_\Omega \frac{1}{\pi\rho} v(x_2, y_2) dx_2 dy_2 && (8.141)\\
= \; & \int_{\{w\in\Omega|\rho<1\}} \frac{1}{\pi\rho} v(x_2, y_2) dx_2 dy_2 + \int_{\{w\in\Omega|1\leq\rho\leq M\}} \frac{1}{\pi\rho} v(x_2, y_2) dx_2 dy_2 \\
& + \int_{\{w\in\Omega|\rho>M\}} \frac{1}{\pi\rho} v(x_2, y_2) dx_2 dy_2 && (8.142)\\
\leq \; & ||v|_{x_2>M}||_{p_1} \frac{1}{\pi} \left\{ \int_0^{2\pi} \int_0^1 \left(\frac{1}{\rho}\right)^{q_1} \rho d\rho d\theta \right\}^{\frac{1}{q_1}} \\
& + ||v|_{x_2>M}||_{p_2} \frac{1}{\pi} \left\{ \int_0^{2\pi} \int_1^M \left(\frac{1}{\rho}\right)^{q_2} \rho d\rho d\theta \right\}^{\frac{1}{q_2}} + \frac{1}{\pi M}||v||_1 && (8.143)\\
= \; & ||v|_{x_2>M}||_{p_1} \frac{1}{\pi} \left\{ 2\pi \int_0^1 \rho^{1-q_1} d\rho \right\}^{\frac{1}{q_1}} \\
& + ||v|_{x_2>M}||_{p_2} \frac{1}{\pi} \left\{ 2\pi \int_1^M \rho^{1-q_2} d\rho \right\}^{\frac{1}{q_2}} + \frac{1}{\pi M}||v||_1 && (8.144)
\end{aligned}$$

$$\leq \kappa_1||v|_{x_2>M}||_{p_1} + \kappa_2||v|_{x_2>M}||_{p_2} + \frac{1}{\pi M}||v||_1 \to 0 \text{ as } M \to \infty \tag{8.145}$$

for some constants κ_1 and κ_2, noting that $\int_0^1 \rho^{1-q_1} d\rho$ and $\int_1^\infty \rho^{1-q_2} d\rho$ are finite by our choice of p_1 and p_2. This completes the proof.

Theorem 2 *Let non-negative $f_0 \in L^1(\Omega) \bigcap L^p(\Omega)$ for $p > 2$. Define*

$$\varphi_\lambda(w) = \frac{1}{2}\int_\Omega wKwd\mu - \lambda \int_\Omega wyd\mu. \tag{8.146}$$

Let $\tilde{f}$ be a maximiser for φ_λ relative to $\overline{R(f_0)}^w$ (we know such a $\tilde{f}$ exists). Then the following are true.

(i) $\tilde{u} = K\tilde{f}$ satisfies

$$-\Delta\tilde{u} = \phi(\tilde{u} - \lambda y) \tag{8.147}$$

in the weak sense, for some increasing function ϕ.

(ii) Except possibly for a set of zero measure,

$$\tilde{f}^{-1}(0,\infty) \subset (K\tilde{f} - \lambda y)^{-1}(0,\infty). \tag{8.148}$$

Then we have

$$\mu(\tilde{f}^{-1}(0,\infty)) \leq \mu((K\tilde{f} - \lambda y)^{-1}(0,\infty)) < \infty \tag{8.149}$$

and in particular if the set $f_0^{-1}(0,\infty)$ has infinite measure, then there exists no maximiser for φ_λ relative to the set of rearrangements.

Further $(K\tilde{f} - \lambda y)^{-1}(0,\infty)$ is a bounded set, whence the set $\tilde{f}^{-1}(0,\infty)$ is bounded.

(iii) Suppose $\tilde{f} \notin R(f_0)$. Then

$$(K\tilde{f} - \lambda y)^{-1}(0,\infty) = \tilde{f}^{-1}(0,\infty) \tag{8.150}$$

except possibly for a set of measure zero, and $(K\tilde{f} - \lambda)^{-1}(0)$ *has zero measure.*

Proof For $\tilde{f} \in R(f_0)$, (i) follows by Lemma 7. For $\tilde{f} \in \overline{R(f_0)}^w \backslash R(f_0)$, $\tilde{f}$ is a maximiser for φ_λ relative to $\overline{R(\tilde{f})}^w$ $(\subset \overline{R(f_0)}^w)$, and $\tilde{f} \in R(\tilde{f})$. We can now apply Lemma 7 to obtain (i).

We consider (ii). (8.148) follows from Lemma 8. Lemma 10 shows the existence of $M > 0$ such that for $|x| > M$,

$$|K\tilde{f}(x,y)| < \lambda y \tag{8.151}$$

whence $K\tilde{f}(x,y) - \lambda y < 0$ for $|x| > M$. Therefore the set $(K\tilde{f} - \lambda y)^{-1}(0,\infty)$ is bounded. Thus $\tilde{f}^{-1}(0,\infty)$ is bounded, and further if $f_0^{-1}(0,\infty)$ has infinite measure, then there exists no maximiser for φ_λ relative to the set of rearrangements.

(iii) follows by Lemma 9 and Corollary 2. This completes the proof.

Theorem 3 *Let non-negative* $f_0 \in L^1(\Omega) \bigcap L^p(\Omega)$, *for* $p > 2$. *Define, for* $\lambda > 0$,

$$\varphi_\lambda(v) = \frac{1}{2}\int_\Omega vKv\,d\mu - \lambda \int_\Omega vy\,d\mu. \tag{8.152}$$

Fix $\lambda > 0$. *Let* $\tilde{f}$ *be a maximiser for* φ_λ *relative to* $\overline{R(f_0)}^w$. *Then there exists a sequence* $\{\xi_n\}_{n=1}^\infty$ *such that* $\xi_n \in R(f_0)$ *for each* $n \in \mathbf{N}$, $\varphi_\lambda(\xi_n) \to \varphi_\lambda(\tilde{f})$, *and* $\{\xi_n\}_{n=1}^\infty$ *has weak limit* $\tilde{f}$ *in* $L^p(\Omega)$.

Proof Theorem 2 (ii) yields that $\tilde{f}^{-1}(0,\infty)$ is bounded. We choose $M > 0$ such that $\tilde{f}^{-1}(0,\infty) \subset [-M, M] \times (0, R) = \tilde{M}$, say. Further, from Theorem 1(ii), we have $\tilde{f} \in RC(f_0)$. We can find $\xi_0 \in R(f_0)$ such that $\xi_0|_{\tilde{M}} = \tilde{f}|_{\tilde{M}}$. Write $M_0 = \Omega \backslash \tilde{M}$. Define, for $n \in \mathbf{N}$,

$$M_n = (-\infty, -M - n) \bigcup (M + n, \infty) \times \left(0, \frac{R}{n}\right). \tag{8.153}$$

For each $n \in \mathbb{N}$, there exists a measure preserving transformation $T_n : M_n \to M_0$ (which exists by Chapter 3, Theorem 1). Define, for each $n \in \mathbb{N}$,

$$\xi_n(z) = \begin{cases} \tilde{f}(z) & z \in \tilde{M} \\ \xi_0 \circ T_n(z) & z \in M_n \\ 0 & z \in \Omega \backslash (\tilde{M} \bigcup M_n) \end{cases}$$

for $z \in \Omega$. The measure preserving properties of T_n ensure $\xi_n \in R(f_0)$ for each $n \in \mathbb{N}$. Let $g \in L^q(\Omega)$, where q denotes the conjugate exponent of p. Then

$$\left| \int_\Omega (\xi_n - \tilde{f}) g d\mu \right| \le ||\xi_0||_p ||g|_{|z|>M+n}||_q \to 0 \tag{8.154}$$

as $n \to \infty$. Thus $\xi_n \xrightarrow{w} \tilde{f}$.

Define, for non-negative $v \in L^1(\Omega) \bigcap L^p(\Omega)$, $p > 2$,

$$\Psi(v) = \frac{1}{2} \int_\Omega v K v d\mu. \tag{8.155}$$

Ψ is (strongly) continuous (using Lemma 1 and Hölder's inequality) and Ψ is strictly convex (from the proof of Corollary 1). Thus Ψ is weakly sequentially lower semicontinuous. Further

$$\left| \int_\Omega \xi_n y d\mu - \int_\Omega \tilde{f} y d\mu \right| = \left| \int_\Omega (\xi_n - \tilde{f}) y d\mu \right| \le \frac{R}{n} ||\xi_0||_1 \to 0 \tag{8.156}$$

as $n \to \infty$. Thus we have

$$\begin{aligned} \varphi_\lambda(\tilde{f}) &= \frac{1}{2} \int_\Omega \tilde{f} K \tilde{f} d\mu - \lambda \int_\Omega \tilde{f} y d\mu && (8.157) \\ &\le \liminf_{n\to\infty} \frac{1}{2} \int_\Omega \xi_n K \xi_n d\mu - \lim_{n\to\infty} \lambda \int_\Omega \xi_n y d\mu && (8.158) \\ &= \liminf_{n\to\infty} \varphi_\lambda(\xi_n). && (8.159) \end{aligned}$$

However $\tilde{f}$ is a maximiser of φ_λ relative to $\overline{R(f_0)}^w$, therefore $\varphi_\lambda(\xi_n) \to \varphi_\lambda(\tilde{f})$ as required. This completes the proof.

Chapter 9

Conclusions and Conjectures

Let non-negative $f_0 \in L^p(\Omega, \mu)$, where Ω is an open unbounded subset of $\mathbf{R}^n$ of infinite μ-measure, where μ is a σ-finite measure absolutely continuous with respect to n-dimensional Lebesgue measure. For $1 < p < \infty$, $\overline{R(f_0)}^w$ was characterised in Chapter 3, and shown to be weakly compact and convex. As regards $p = 1$, in Chapter 4 we showed that

$$\overline{R(f_0)}^w \subset \left\{ w \geq 0 \middle| \int_\Omega (w - \sigma)_+ d\mu \leq \int_\Omega (f_0 - \sigma)_+ d\mu, \forall \sigma > 0, \text{ and } ||w||_1 = ||f_0||_1 \right\} \tag{9.1}$$

The question whether the inclusion in (9.1) is strict remains open. We expect the above results to be applicable to the study of variational problems over the set of rearrangement of some fixed function.

We consider the variational problem discussed in Chapters 6 and 7 (similar remarks can equally well be made concerning the variational problem of Chapter 8). For fixed $\lambda > 0$, we recall

$$\Psi_\lambda(v) = \frac{1}{2} \int_\Omega vKv d\nu - \frac{1}{2} \lambda \int_\Omega vy^2 d\nu \tag{9.2}$$

where the appropriate definitions may be found in Chapter 6. For non-negative

$f_0 \in L^1(\Omega)\bigcap L^p(\Omega)$, $p > \frac{5}{2}$, we know that Ψ_λ attains a maximum relative to $\overline{R(f_0)}^w$. More precisely, we know there is a Steiner symmetric maximiser. It is of interest to determine whether maximisers which are not Steiner symmetric (up to translation) exist.

The proof of Lemma 3 of Chapter 6 yields that for non-negative $w \in L^1(\Omega) \bigcap L^p(\Omega)$, p as above,

$$\begin{aligned} & \Psi_\lambda(w^*) - \Psi_\lambda(w) && (9.3)\\ \geq\ & \int_\Omega w^*(Kw)^* d\nu - \frac{1}{2}||(Kw)^*||_H^2 + \frac{1}{2}||Kw||_H^2 - \int_\Omega wKwd\nu && (9.4)\\ \geq\ & \frac{1}{2}||Kw||_H^2 - \frac{1}{2}||(Kw)^*||_H^2 && (9.5)\end{aligned}$$

where we have have used inequality (6.21). (6.22) yields that

$$||(Kw)^*||_H \leq ||Kw||_H \tag{9.6}$$

If the inequality in (9.6) is strict, we have $\Psi_\lambda(w^*) > \Psi_\lambda(w)$, whence w is not a maximiser. [**5**] shows that for non-negative $u \in H$ with compact support and absolutely continuous distribution function, we have

$$||u^*||_H < ||u||_H \tag{9.7}$$

unless u is almost everywhere equal to a translate of u^*. In view of this, we conjecture that any maximiser is Steiner symmetric up to translation.

For every maximiser $\tilde{f}$ relative to $\overline{R(f_0)}^w$, $\tilde{f} \in RC(f_0)$, and $\tilde{u} = K\tilde{f}$ satisfies

$$\mathcal{L}\tilde{u} = \phi(\tilde{u} - \frac{1}{2}\lambda y^2) \tag{9.8}$$

in the weak sense for some increasing function ϕ. (9.8) is the equation for the stream function of a steady flow. We seek to show that (9.8) holds almost everywhere in Ω, using techniques discussed in [**9**]. Further we have

$$\tilde{f}^{-1}(0,\infty) \subset (K\tilde{f} - \frac{1}{2}\lambda y^2)^{-1}(0,\infty) \tag{9.9}$$

except possibly for a set of measure zero, and if $\tilde{f} \notin R(f_0)$, (9.9) holds with equality (except possibly for a set of measure zero). We have seen that $(K\tilde{f} - \frac{1}{2}\lambda y^2)^{-1}(0,\infty)$ is a bounded set. For a given $\lambda > 0$, let $\tilde{f}_\lambda$ denote a maximiser of Ψ_λ relative to $\overline{R(f_0)}^w$. We have seen that $\tilde{f}_\lambda \in RC(f_0)$, therefore there exists an extended real number l_λ such that $\tilde{f}_\lambda|_{\tilde{f}_\lambda^{-1}(l_\lambda,\infty)}$ is a rearrangement of $f_0|_{f_0^{-1}(l_\lambda,\infty)}$, $\nu\left(f_0^{-1}(l_\lambda)\right) \geq \nu\left(\tilde{f}_\lambda^{-1}(l_\lambda)\right)$, and $\tilde{f}_\lambda^{-1}(0,l_\lambda)$ has zero measure. The upper bound we obtained for the operator Kw (w as above) in Lemma 5 of Chapter 7 suggests the conjecture that for $0 < \lambda_1 < \lambda_2$ we have

$$\nu\left((K\tilde{f}_{\lambda_2} - \frac{1}{2}\lambda_2 y^2)^{-1}(0,\infty)\right) \leq \nu\left((K\tilde{f}_{\lambda_1} - \frac{1}{2}\lambda_1 y^2)^{-1}(0,\infty)\right). \tag{9.10}$$

If (9.10) does hold, then we have

$$\nu\left(\tilde{f}_{\lambda_2}^{-1}(0,\infty)\right) \leq \nu\left(\tilde{f}_{\lambda_1}^{-1}(0,\infty)\right). \tag{9.11}$$

By way of explanation, we note that (9.11) holds trivially if $\tilde{f}_{\lambda_1} \in R(f_0)$, otherwise (9.9) (with equality except possibly for a set of measure zero for $\tilde{f}_{\lambda_1}$) and (9.10) yield (9.11). (9.11) shows that $\tilde{f}_{\lambda_2} \in R(\tilde{f}_{\lambda_1})$. Moreover (9.11) implies that $l_{\lambda_1} \leq l_{\lambda_2}$, that is the maximisers $\tilde{f}_\lambda$ become "more curtailed" as λ increases. We conjecture that l_λ depends continuously on λ.

Finally we note that for each maximiser $\tilde{f}$, there is a weakly converging sequence $\{f_n\}_{n=1}^\infty$ in $R(f_0)$ with weak limit $\tilde{f}$ (in $L^p(\Omega)$) such that $\Psi_\lambda(f_n) \to \Psi_\lambda(\tilde{f})$. However we note that there exist maximising sequences (relative to $\overline{R(f_0)}^w$) where the weak subsequential limit (which exists because $\overline{R(f_0)}^w$ is weakly sequentially compact) is not a maximiser. For a given maximiser, we can choose a sequence of translates of the maximiser which converges weakly to zero in $L^p(\Omega)$.

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Appendix

Lemma 1 *Let $q \geq 2$. The integral $\int_0^1 |(\log \rho)|^q \rho d\rho$ is finite.*

Proof We have that

$$\int_0^1 |(\log \rho)|^q \rho d\rho = \int_{-\infty}^0 |u|^q \exp(2u) du \tag{9.12}$$
$$= \int_{-\infty}^{-1} |u|^q \exp(2u) du + \int_{-1}^0 |u|^q \exp(2u) du \tag{9.13}$$
$$\leq \int_{-\infty}^{-1} u^{\tilde{q}} \exp(2u) du + 1 \tag{9.14}$$

where $\tilde{q}$ is an even integer greater than or equal to q, and we have used the fact that $\int_{-1}^0 |u|^q \exp(2u) du \leq 1$. We may integrate $\int_{-\infty}^{-1} u^{\tilde{q}} \exp(2u) du$ by parts to obtain a finite bound for the original integral . This completes the proof.

Lemma 2 *Let Ω be a bounded domain in $\mathbf{R}^n$. Let $u \in W^{1,2}(\Omega)$, $u^+ \in H_0^1(\Omega)$ (where $u^+(x) = \max\{u(x), 0\}$). Let $v \in W^{1,2}(\Omega)$, $v^- \in H_0^1(\Omega)$ (where $v^-(x) = -\min\{u(x), 0\}$). Then $(v-u)^- \in H_0^1(\Omega)$.*

Proof We note that

$$0 \leq (v-u)^- \leq v^- + u^+ \tag{9.15}$$

and $v - u \in W^{1,2}(\Omega)$, whence $(v-u)^- \in W^{1,2}(\Omega)$, and $v^- + u^+ \in H_0^1(\Omega)$.

[**20**, Lemma 7.6, page 152] yields that for $u \in W^{1,2}(\Omega)$, $|u| \in W^{1,2}(\Omega)$, and moreover the map $\Psi(u) = |u|$ is continuous on $W^{1,2}(\Omega)$. Let $\{f_n\}_{n=1}^{\infty} \subset C^{\infty}(\Omega)$ be such that f_n is non–negative for each $n \in \mathbf{N}$, and $f_n \to (v-u)^-$ in $W^{1,2}(\Omega)$. (We may choose f_n non–negative without loss of generality). Let $\{g_n\}_{n=1}^{\infty} \subset C_0^{\infty}(\Omega)$ be such that g_n is non–negative for each $n \in \mathbf{N}$, and $g_n \to v^- + u^+$ in $H_0^1(\Omega)$. (We may choose g_n non–negative without loss of generality). Define $\varphi_n(x) = \min\{f_n(x), g_n(x)\}$. For each $n \in \mathbf{N}$, supp $\varphi_n \subset$ supp g_n, thus φ_n has compact support. Now we may write

$$\varphi_n(x) = \frac{1}{2}\{f_n(x) + g_n(x) - |f_n(x) - g_n(x)|\}. \tag{9.16}$$

The above remarks yield that $\varphi_n \in W^{1,2}(\Omega)$, and φ_n has compact support, therefore $\varphi_n \in H_0^1(\Omega)$. The continuity of the map $\Psi(u) = |u|$ yields that

$$\varphi_n \to \frac{1}{2}\Big\{(v-u)^- + (v^- + u^+) - |(v-u)^- - (v^- + u^+)|\Big\} = (v-u)^- \tag{9.17}$$

in $W^{1,2}(\Omega)$, whence $(v-u)^- \in H_0^1(\Omega)$. This completes the proof.